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Analysis of a semi-augmented mixed finite element method for double-diffusive natural convection in porous media ☆

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ABSTRACT

In this paper we study a stationary double-diffusive natural convection problem in porous media given by a Navier-Stokes/Brinkman type system, for describing the velocity and the pressure, coupled to a vector advection-diffusion equation relate to the heat and substance concentration, of a viscous fluid in a porous media with physical boundary conditions. The model problem is rewritten in terms of a first-order system, without the pressure, based on the introduction of the strain tensor and a nonlinear pseudo-stress tensor in the fluid equations. After a variational approach, the resulting weak model is then augmented using appropriate redundant penalization terms for the fluid equations along with a standard primal formulation for the heat and substance concentration. Then, it is rewritten as an equivalent fixed-point problem. Well-posedness results for both the continuous and the discrete schemes are stated, as well as the respective convergence result under certain regularity assumptions combined with the Lax-Milgram theorem, and the Banach and Brouwer fixed-point theorems. In particular, Raviart-Thomas elements of order k are used for approximating the pseudo-stress tensor, piecewise polynomials of degree $\leq k$ and $\leq k+1$ are utilized for approximating the strain tensor and the velocity, respectively, and the heat and substance concentration are approximated by means of Lagrange finite elements of order $\leq k+1$. Optimal a priori error estimates are derived and confirmed through some numerical examples that illustrate the performance of the proposed semi-augmented mixed-primal scheme.

1. Introduction

In nature and several technological applications, transport phenomena widely occur as a result of a combined heat and mass transfer that are driven by buoyancy effects due to both temperature and concentration variations (see, e.g., [14,32,34]). Such processes, also known as thermosolutal or double-diffusive natural convection, involving fluid circulation in a porous media, are frequently found in astrophysics, oceanology, metallurgy, electrophysics and geophysics, but also appear in several engineering applications such as filtration processes, geothermal energy exploitation, spreading on porous substrates, bio-film growth, gasification of biomass, to name a few.

From the mathematical point of view, the Darcy-Oberbeck-Boussinesq model allows to adequately describe and quantify this complex flow by means of a nonlinear partial differential equations system. More precisely, the momentum and conservation of fluid mass give rise to a Navier-Stokes/Darcy type system for describing the fluid flow in the porous media which, in turn, is coupled via buoyancy forces and convective mass and heat transfer to a vector advection-diffusion equation for describing the substance concentration and the temperature, as a result of an energy and mass transfer balance (see, e.g., [32,34]).

Many computational techniques have been developed so far in order to numerically solve and simulate this problem and related ones (see [1,3,10,13,16,17,21,25,26,28,29,33,37], and the references therein). Particularly, the contributions [1,3,33,25,26,28,37] deal with double-diffusive convection in a cavity, whereas in [10,16,17] the authors consider the phenomenon in a porous media.

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In [33], a finite volume method is proposed and applied to solve agro-food processes, whereas some methods based on finite elements for this problem are [1,3]. In [1], the authors proposed stabilized finite element formulations based on the SUPG (Streamline-Upwind/Petrov-Galerkin) and PSPG (Pressure-Stabilization/Petrov-Galerkin) methods to solve the problem in unsteady state. Numerical simulations in two and three dimensions illustrate the accuracy and performance of this technique. However, the theoretical analyses of the associate continuous and the discrete variational problems as well as the convergence of the method are not carried out there, and the method only allows to carry out low-order approximations of the main unknowns. On the other hand, in [3] the problem is considered in steady state and analyzed by using the boundary control theory. The authors formulate and prove solvability results for the corresponding boundary control problem, state local uniqueness and stability of optimal solutions.

Focusing in double-diffusive viscous flow in porous media, [16] proposes a technique consisting of a projection-based stabilization method in the unsteady state. There, the convergence of the velocity, temperature and concentration in the semi-discrete case is derived. In addition, some numerical experiments are reported to confirm optimal order error estimates and to compare their results with previous ones. In [10] the authors construct a divergence-conforming primal scheme and establish the existence and uniqueness results for the continuous problem and the discrete scheme as well as convergence properties. On the other hand, in [17] the authors propose a high-order fully-mixed method based on the introduction of the velocity gradient, the temperature gradient and the concentration gradient as new unknowns into the problem. The resulting formulation has a saddle-point structure on reflexive Banach spaces for both the Navier-Stokes/Darcy and the thermal energy conservation equations. There, it is particularly shown that the discrete scheme is well-posed and an a priori error estimate for the Galerkin scheme is also derived under sufficiently small data. However, feasible finite element subspaces must be constructed over meshes with a macro-element structure in order to satisfy an inf-sup compatibility condition, which in turn significantly increased the computational cost, especially in the three-dimensional case.

According to the above discussion and in order to contribute with the design and analysis of new mixed finite schemes for simulating double-diffusive convection in porous media, we now propose a new semi-augmented mixed finite element method in which the strain tensor and a pseudo-stress tensor are introduced as primary unknowns of interest in the fluid equations and the pressure is eliminated from the system by its own definition. To avoid any inf-sup restriction, to guarantee greater flexibility regarding finite element spaces and lower computational cost than [17], the respective variational formulation is augmented by using appropriate redundant penalization Galerkin type-terms based on the constitutive and equilibrium equations combined with a primal formulation for the heat and substance concentration equations in standard Hilbert spaces. In this way, the aforementioned strain tensor, the nonlinear pseudostress, the velocity and a vector field whose components are the temperature of the fluid and the substance concentration are the main unknowns of the resulting coupled system. Moreover, physical boundary conditions are considered. Indeed, a no-slip condition (that is, homogeneous Dirichlet condition) for the fluid velocity, a prescribed temperature and substance concentration on a Dirichlet boundary and no heat/mass flow across an isolated surface/homogeneous Neumann condition.

Concerning the solvability analysis, we proceed similarly to [4,18] by introducing an equivalent fixed-point setting. According to this, combining the Lax-Milgram theorem with the classical Banach and Brouwer fixed-point theorems, we establish the respective solvability of the continuous problem and the associated Galerkin scheme, under suitable regularity assumptions (see [7], for further details), a feasible choice of parameters and, in the discrete case, for any family of finite element subspaces. To handle the non-homogeneous Dirichlet condition for the temperature and concentration, we carry out a rigorous treatment of the boundary data throughout the analysis by means of appropriate extensions involving the Scott-Zhang interpolator (in the discrete case), which allows us to establish the well-posedness of our scheme, along with its convergence result and the respective a priori error bounds. Up to the best of our knowledge, because of the difficulties that can arise in the analysis, the physically relevant non-homogeneous Dirichlet condition case is usually either omitted or not considered; this is what motivates us to contribute in this direction as well.

A Strang-type lemma, enables us to derive the corresponding Céa estimate and to provide optimal a priori error bounds for the Galerkin solution. In turn, the pressure can be recovered by a post-processed of the discrete solutions, preserving the same rate of convergence. Finally, some numerical experiments are presented to illustrate the performance of the technique, including the unsteady case with unknown solution, to confirm the expected orders.

The contents of this paper are presented as follows. At the end of this section, we introduce some standard notations to be used throughout the manuscript. In Section 2, we introduce the model problem and the data. We also derive an equivalent first-order equations in terms of additional variables. Then, in Section 3, we derive the semi-augmented mixed-primal variational formulation and establish its well-posedness. The associated Galerkin scheme is introduced and analyzed in Section 4. In Section 5, we derive the corresponding Céa estimate and, finally, in Section 6 we present some numerical examples illustrating the performance of our semi-augmented mixed-primal finite element method.

1.1. Notations

Let us denote by $\Omega \subseteq \mathrm{R}^n$, with $n \in \{2,3\}$, a given bounded domain with polygonal/polyhedral boundary Γ with outward unit normal vector $\boldsymbol{n}$ and let $\Gamma_{\mathrm{D}}, \Gamma_{\mathrm{N}} \subseteq \Gamma$ be such that $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, $|\Gamma_{\mathrm{D}}| > 0$ and $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ with norm $\|\cdot\|_{0,p,\Omega}$ or $\|\cdot\|_{0,\Omega}$ if $p=2$, and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$, and semi-norm $|\cdot|_{s,\Omega}$. In particular, when A denotes a generic scalar functional space, then we will denote its vectorial and tensor counterparts by $\mathbf{A}$ and $\mathbb{A}$, respectively.

For vector fields $\boldsymbol{v} = (v_i)_{1\le i\le n}$ and $\boldsymbol{w} = (w_i)_{1\le i\le n}$, we set the gradient, divergence and dyadic product operators, as

$$\nabla \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{1\le i,j\le n}, \quad \operatorname{div} \boldsymbol{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \boldsymbol{v}\otimes\boldsymbol{w} := (v_i w_j)_{1\le i,j\le n},$$

respectively. Furthermore, given the tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{1\le i,j\le n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{1\le i,j\le n}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{1\le i,j\le n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{j=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. We recall that the space

$$\mathbb{H}(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \right\},$$

equipped with the usual norm

$$\|\tau\|^2_{\mathbf{div},\Omega} := \|\tau\|^2_{0,\Omega} + \|\mathbf{div}\,\tau\|^2_{0,\Omega},$$

is a Hilbert space. Finally, we employ $\mathbf{0}$ to denote a generic null vector and use c or C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2. The model problem

This section introduces the mathematical model we address in the present work as well as the auxiliary unknowns that are introduced and considered in the subsequent variational formulation. Under the Oberbeck-Boussinesq approximation framework, double-diffusive natural convection phenomenon in a porous media is described in terms of a Navier-Stokes/Brinkman model coupled to a system of diffusion-advection equations. In the stationary state, the problem consists of finding the velocity $\boldsymbol{u}$, the pressure p, and the vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^{\mathrm{t}}$ when $n = 2$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, 0)^{\mathrm{t}}$ when $n = 3$, where φ_1 and φ_2 are the temperature and the concentration fields, respectively, of a fluid in a confined porous enclosure Ω, satisfying the set of equations:

$$\begin{aligned} \gamma\,\boldsymbol{u} - 2\,\mathbf{div}\,(\nu(\boldsymbol{\varphi})\mathbf{e}(\boldsymbol{u})) + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla\,p &= (\boldsymbol{\alpha}\cdot\boldsymbol{\varphi})\,\boldsymbol{g} \quad \text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \\ -\mathbf{div}(\mathbb{K}\nabla\boldsymbol{\varphi}) + (\nabla\boldsymbol{\varphi})\boldsymbol{u} &= \mathbf{0} \quad \text{in} \quad \Omega, \end{aligned} \tag{2.1}$$

where $\mathbf{e}(\boldsymbol{u})$ stands for the symmetric part of the velocity gradient, i.e., $\mathbf{e}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{t}})$. The data are the gravitational force $\boldsymbol{g}$, the positive constant γ corresponding to the reciprocal of the Darcy number, the (thermal and solute) expansion coefficients vector $\boldsymbol{\alpha}$, and the diagonal matrix of (thermal and mass) diffusion constants $\mathbb{K} = \operatorname{diag}(k_i)_{1\le i\le n} \in \mathbb{L}^\infty(\Omega)$, with $k_i = 0$ when $i = 3$, which is assumed to be a uniformly positive definite tensor, which means that there exists a positive constant k_0 such that

$$(\mathbb{K}\,\boldsymbol{x})\cdot\boldsymbol{x} \ge k_0\,|\boldsymbol{x}|^2 \qquad \forall\,\boldsymbol{x}\in\mathrm{R}^n, \quad \text{with} \quad k_0 = \min\{k_1, k_2, \dots, k_n\}. \tag{2.2}$$

In turn, the kinematic viscosity $\nu : \mathrm{R}^+\times\mathrm{R}^+ \longrightarrow \mathrm{R}^+$ is assumed to be a bounded and Lipschitz continuous function that might depend on both the temperature and the mass concentration. That is, we assume the existence of positive constants ν_1, ν_2, and L_ν that satisfy

$$\nu_1 \le \nu(\boldsymbol{\phi}) \le \nu_2 \quad \text{and} \quad |\nu(\boldsymbol{\phi}) - \nu(\boldsymbol{\psi})| \le L_\nu\,|\boldsymbol{\phi} - \boldsymbol{\psi}| \qquad \forall\,\boldsymbol{\phi},\boldsymbol{\psi}\in\mathrm{R}\times\mathrm{R}^+. \tag{2.3}$$

The system (2.1) is finally completed with a non-slip condition on the whole boundary for the velocity and physical boundary conditions for both the temperature and the concentration fields, that is

$$\boldsymbol{u} = \mathbf{0} \quad \text{on}\,\Gamma, \qquad \boldsymbol{\varphi} = \boldsymbol{\varphi}_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}} \qquad \text{and} \qquad (\nabla\boldsymbol{\varphi})\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_{\mathrm{N}}, \tag{2.4}$$

where $\boldsymbol{\varphi}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$ is a known trace function on Γ_{D}.

Next, proceeding similarly to [12], we introduce the strain tensor $\boldsymbol{t} := \mathbf{e}(\boldsymbol{u})$ as an auxiliary variable and then define

$$\boldsymbol{\sigma} := 2\nu(\boldsymbol{\varphi})\boldsymbol{t} - (\boldsymbol{u}\otimes\boldsymbol{u}) - \left(p + c_{\boldsymbol{u}}\right)\mathbb{I} \quad \text{in} \quad \Omega, \tag{2.5}$$

as an additional tensorial unknown, where $c_{\boldsymbol{u}}$ is a constant to be suitably defined next (see equation (2.7) below). Thus, noting that the incompressibility condition of the fluid ($\operatorname{div}\boldsymbol{u} = 0$) implies that $\mathbf{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, we get from the first relation of (2.1) the equilibrium equation

$$\gamma\boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma} = (\boldsymbol{\alpha}\cdot\boldsymbol{\varphi})\,\boldsymbol{g} \quad \text{in} \quad \Omega,$$

and taking deviatoric part in (2.5), we find that the constitutive relation defined by (2.5) can be written as

$$\boldsymbol{\sigma}^{\mathrm{d}} = 2\nu(\boldsymbol{\varphi})\boldsymbol{t} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} \quad \text{in} \quad \Omega$$

where $\boldsymbol{t}^{\mathrm{d}} = \boldsymbol{t}$ since $\operatorname{div}\boldsymbol{u} = 0$. Thus, the pressure is eliminated from the system, however taking trace in (2.5), we find that it can be easily recovered according to the formula

$$p = -\frac{1}{n}\Big[\operatorname{tr}(\boldsymbol{\sigma} + (\boldsymbol{u}\otimes\boldsymbol{u}))\Big] - c_{\boldsymbol{u}}, \qquad \text{in} \qquad \Omega. \tag{2.6}$$

Now, since $p \in \mathrm{L}^2_0(\Omega)$ (which as usual is clearly required for uniqueness of an eventual pressure solution to (2.1)), the equation (2.6) suggests to define

$$c_{\boldsymbol{u}} = -\frac{1}{n|\Omega|}\int_\Omega \operatorname{tr}(\boldsymbol{u}\otimes\boldsymbol{u}) = -\frac{1}{n|\Omega|}\|\boldsymbol{u}\|^2_{0,\Omega}, \tag{2.7}$$

so as to get the equivalence

$$\int_\Omega p = 0 \qquad \text{if, and only if,} \qquad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0. \tag{2.8}$$

According to the above discussion, the system (2.1) and (2.4) equivalently reads: Find $\boldsymbol{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varphi}$ such that

$$\begin{gathered} 2\nu(\boldsymbol{\varphi})\boldsymbol{t} - \boldsymbol{\sigma}^{\mathrm{d}} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} = \mathbf{0} \quad \text{in} \quad \Omega, \quad \gamma\boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma} = (\boldsymbol{\alpha}\cdot\boldsymbol{\varphi})\,\boldsymbol{g} \quad \text{in} \quad \Omega, \\ \boldsymbol{t} + \boldsymbol{\omega}(\boldsymbol{u}) = \nabla\boldsymbol{u} \quad \text{in} \quad \Omega, \quad -\mathbf{div}(\mathbb{K}\nabla\boldsymbol{\varphi}) + (\nabla\boldsymbol{\varphi})\boldsymbol{u} = \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \quad \boldsymbol{\varphi} = \boldsymbol{\varphi}_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}}, \quad (\nabla\boldsymbol{\varphi})\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{N}}, \quad \text{and} \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0, \end{gathered} \tag{2.9}$$

where $\boldsymbol{\omega}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathrm{t}})$ is the skew-symmetric part of the velocity gradient. We emphasize that the introduction of the variables $\boldsymbol{t}$ and $\boldsymbol{\sigma}$ as new unknowns in the system allows us to equivalently rewrite the Navier-Stokes-Brinkman model (first row of (2.1)) in terms of a first-order set of equations. Also, observe that according to (2.6)-(2.8), the zero mean value restriction of the pressure on the domain is imposed via the respective restriction on $\operatorname{tr}\boldsymbol{\sigma}$ in the last equation of (2.9). Note further that the incompressibility condition of the fluid is implicitly present via the equilibrium relation defined by $\boldsymbol{\sigma}$ according to the second equation in the first row of (2.9).

Remark 2.1. It is important to mention here, that we follow [12] by introducing the auxiliary unknown $\boldsymbol{t} := \mathbf{e}(\boldsymbol{u})$. Using this we can obtain below a way to establish the solvability analysis of a continuous formulation. Furthermore, this suggests that the introduction of the vorticity $\boldsymbol{\omega}(\boldsymbol{u}) = \nabla\boldsymbol{u} - \boldsymbol{t}$, as an auxiliary unknown, is unnecessary since the symmetry of the pseudostress tensor $\boldsymbol{\sigma}$ can be imposed in an ultra-weak sense (see, e.g., [5,8]).

3. The continuous formulation

In this section we introduce and analyze the weak formulation proposed for the system described by (2.9). In Section 3.1 we firstly deduce an augmented mixed variational formulation of (2.9) and then we rewrite it as a fixed-point problem in Section 3.2, whose analysis is addressed through Sections 3.3 and 3.4.

3.1. The semi-augmented mixed-primal variational formulation

To begin with, the fact that the trace of the tensor solution $\boldsymbol{\sigma}$ of system (2.9) has zero mean value in Ω (see last equation of (2.9)) suggests to introduce the space

$$\mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0 \right\}.$$

In addition, by their own definitions, we introduce the following space for the strain tensor $\boldsymbol{t}$, as

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \left\{ \boldsymbol{r} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{r}^{\mathrm{t}} = \boldsymbol{r} \quad \text{and} \quad \operatorname{tr}(\boldsymbol{r}) = 0 \right\}.$$

In turn, because of the boundary conditions for the temperature and concentrations (see second and third equations of last row in (2.9)) we consider the closed subspace

$$\mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) := \left\{ \boldsymbol{\psi} \in \mathbf{H}^1(\Omega) : \quad \boldsymbol{\psi}\Big|_{\Gamma_{\mathrm{D}}} = \mathbf{0} \right\},$$

for which, from the generalized Poincaré inequality, we know that there exists $c_{\mathrm{gp}} > 0$, depending only on Ω and Γ_{D}, such that

$$\|\boldsymbol{\psi}\|_{1,\Omega} \le c_{\mathrm{gp}} |\boldsymbol{\psi}|_{1,\Omega} \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{3.1}$$

Now, testing the first equation from first row in (2.9) with $\boldsymbol{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, we obtain

$$2\int_\Omega \nu(\boldsymbol{\varphi})\boldsymbol{t} : \boldsymbol{r} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{r} - \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} : \boldsymbol{r} = 0 \qquad \forall\, \boldsymbol{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega). \tag{3.2}$$

At this point, we readily note that in order to bound the third term on the left hand side of (3.2), and thanks to the continuous injection $\boldsymbol{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ (see e.g. [2] or [35]), we require the unknown $\boldsymbol{u}$ to live in $\mathbf{H}^1_0(\Omega) := \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{v}|_\Gamma = \mathbf{0}\}$. Indeed, by applying Cauchy-Schwarz and Hölder inequalities, we deduce that there exists a positive constant $c_1(\Omega) := \|\boldsymbol{i}\|^2$, such that

$$\left| \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{w})^{\mathrm{d}} : \boldsymbol{r} \right| \le c_1(\Omega)\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{r}\|_{0,\Omega} \quad \forall\, \boldsymbol{u},\boldsymbol{w} \in \mathbf{H}^1_0(\Omega), \qquad \forall\, \boldsymbol{r} \in \mathbb{L}^2(\Omega). \tag{3.3}$$

Next, multiplying the first equation from second row in (2.9) by a test function $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)$, integrating by parts, using the Dirichlet condition for $\boldsymbol{u}$, and the identity $\boldsymbol{t} : \boldsymbol{\tau} = \boldsymbol{t} : \boldsymbol{\tau}^{\mathrm{d}}$ (since $\boldsymbol{t}$ is trace-free), we get

$$\int_\Omega \boldsymbol{t} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \boldsymbol{\omega}(\boldsymbol{u}) : \boldsymbol{\tau} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega).$$

Likewise, the equilibrium relation associated to $\boldsymbol{\sigma}$ (second equation from first row in (2.9)) is written as

$$\gamma\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_\Omega (\boldsymbol{\alpha}\cdot\boldsymbol{\varphi})\boldsymbol{g}\cdot\boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{H}^1_0(\Omega),$$

whereas, the symmetry of the pseudo-stress tensor is imposed in an ultra-weak sense (see e.g. [8]) through the identity

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\omega}(\boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \mathbf{H}^1_0(\Omega).$$

As for the equation associated to the temperature and concentration (second equation from second row in (2.9)), we simply multiply it by a test function $\boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and, after integrating by parts, and employing the Neumann boundary conditions on Γ_{N}, we find

$$\int_\Omega \mathbb{K}\nabla\boldsymbol{\varphi} : \nabla\boldsymbol{\psi} \;+\; \int_\Omega (\nabla\boldsymbol{\varphi})\boldsymbol{u}\cdot\boldsymbol{\psi} \;=\; 0 \qquad \forall\, \boldsymbol{\psi}\in \mathbf{H}^1_{\Gamma_\mathrm{D}}(\Omega).$$

In this way, a preliminary weak formulation for the coupled problem (2.9) takes the form: Find $(\boldsymbol{t},\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\varphi}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1(\Omega)$, with $\boldsymbol{\varphi}\big|_{\Gamma_\mathrm{D}} = \boldsymbol{\varphi}_\mathrm{D}$, such that

$$\begin{aligned}
2\int_\Omega \nu(\boldsymbol{\varphi})\boldsymbol{t} : \boldsymbol{r} - \int_\Omega \boldsymbol{\sigma}^\mathrm{d} : \boldsymbol{r} - \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^\mathrm{d} : \boldsymbol{r} &= 0 && \forall\,\boldsymbol{r}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega),\\
\int_\Omega \boldsymbol{t} : \boldsymbol{\tau}^\mathrm{d} + \int_\Omega \boldsymbol{\omega}(\boldsymbol{u}) : \boldsymbol{\tau} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} &= 0 && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
-\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\omega}(\boldsymbol{v}) : \boldsymbol{\sigma} + \gamma\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} &= \int_\Omega (\boldsymbol{\alpha}\cdot\boldsymbol{\varphi})\boldsymbol{g}\cdot\boldsymbol{v} && \forall\,\boldsymbol{v}\in\mathbf{H}^1_0(\Omega),\\
\int_\Omega \mathbb{K}\nabla\boldsymbol{\varphi} : \nabla\boldsymbol{\psi} + \int_\Omega (\nabla\boldsymbol{\varphi})\boldsymbol{u}\cdot\boldsymbol{\psi} &= 0 && \forall\,\boldsymbol{\psi}\in\mathbf{H}^1_{\Gamma_\mathrm{D}}(\Omega).
\end{aligned} \tag{3.4}$$

In order to analyze the variational formulation (3.4), and similarly as in [12, Section 2] (see also [4,8,20]), we additionally augment (3.4) by incorporating the following residual Galerkin type-terms coming from the constitutive and equilibrium equations for the fluid,

$$\begin{aligned}
\kappa_1\int_\Omega (\mathbf{e}(\boldsymbol{u}) - \boldsymbol{t}) : \mathbf{e}(\boldsymbol{v}) &= 0 && \forall\,\boldsymbol{v}\in\mathbf{H}^1_0(\Omega),\\
-\kappa_2\gamma\int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \kappa_2\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} &= -\kappa_2\int_\Omega (\boldsymbol{\alpha}\cdot\boldsymbol{\varphi})\boldsymbol{g}\cdot\mathbf{div}\,\boldsymbol{\tau} && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
-\kappa_3\int_\Omega \left\{2\nu(\boldsymbol{\varphi})\boldsymbol{t} - \boldsymbol{\sigma}^\mathrm{d} - (\boldsymbol{u}\otimes\boldsymbol{u})^\mathrm{d}\right\} : \boldsymbol{\tau}^\mathrm{d} &= 0 && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),
\end{aligned} \tag{3.5}$$

where $(\kappa_1,\kappa_2,\kappa_3)$ is a vector of positive parameters to be specified later in Section 3.3.

Hence, letting

$$\vec{\boldsymbol{\xi}} \;:=\; (\boldsymbol{t},\boldsymbol{\sigma},\boldsymbol{u}) \in \mathbb{H} := \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1_0(\Omega),$$

where $\mathbb{H}$ is endowed with the natural norm

$$\|\vec{\boldsymbol{\eta}}\|_{\mathbb{H}} := \left\{ \|\boldsymbol{r}\|^2_{0,\Omega} + \|\boldsymbol{\tau}\|^2_{\mathbf{div},\Omega} + \|\boldsymbol{v}\|^2_{1,\Omega}\right\}^{1/2}, \quad \forall\,\vec{\boldsymbol{\eta}} := (\boldsymbol{r},\boldsymbol{\tau},\boldsymbol{v})\in\mathbb{H},$$

and adding up (3.4) with (3.5), we arrive at the following semi-augmented mixed-primal formulation for the double-diffusive natural convection problem in porous media: Find $(\vec{\boldsymbol{\xi}},\boldsymbol{\varphi})\in\mathbb{H}\times\mathbf{H}^1(\Omega)$, with $\boldsymbol{\varphi}\big|_{\Gamma_\mathrm{D}} = \boldsymbol{\varphi}_\mathrm{D}$, such that

$$\mathbf{A}_{\boldsymbol{\varphi}}(\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\eta}}) + \mathbf{B}_{\boldsymbol{u}}(\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\eta}}) = F_{\boldsymbol{\varphi}}(\vec{\boldsymbol{\eta}}) \qquad \forall\,\vec{\boldsymbol{\eta}}\in\mathbb{H}, \tag{3.6a}$$

$$\mathbf{C}(\boldsymbol{\varphi},\boldsymbol{\psi}) = G_{\boldsymbol{u},\boldsymbol{\varphi}}(\boldsymbol{\psi}) \qquad \forall\,\boldsymbol{\psi}\in\mathbf{H}^1_{\Gamma_\mathrm{D}}(\Omega), \tag{3.6b}$$

where, given $\boldsymbol{\phi}\in\mathbf{H}^1(\Omega)$ and $\boldsymbol{w}\in\mathbf{H}^1_0(\Omega)$, $\mathbf{A}_{\boldsymbol{\phi}}$, $\mathbf{B}_{\boldsymbol{w}}$ and $\mathbf{C}$ are the bilinear forms defined, respectively, as

$$\begin{aligned}
\mathbf{A}_{\boldsymbol{\phi}}(\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\eta}}) \;:=\;& 2\int_\Omega \nu(\boldsymbol{\phi})\boldsymbol{t} : \left\{\boldsymbol{r} - \kappa_3\boldsymbol{\tau}^\mathrm{d}\right\} + \int_\Omega \boldsymbol{t} : \{\boldsymbol{\tau}^\mathrm{d} - \kappa_1\mathbf{e}(\boldsymbol{v})\} - \int_\Omega \boldsymbol{\sigma}^\mathrm{d} : \left\{\boldsymbol{r} - \kappa_3\boldsymbol{\tau}^\mathrm{d}\right\}\\
&+ (1-\kappa_2\gamma)\int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} + \int_\Omega \boldsymbol{\omega}(\boldsymbol{u}) : \boldsymbol{\tau} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\omega}(\boldsymbol{v})\\
&+ \gamma\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \kappa_2\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \kappa_1\int_\Omega \mathbf{e}(\boldsymbol{u}) : \mathbf{e}(\boldsymbol{v}),
\end{aligned} \tag{3.7}$$

and

$$\mathbf{B}_{\boldsymbol{w}}(\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\eta}}) \;:=\; \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{w})^\mathrm{d} : \left\{\kappa_3\boldsymbol{\tau}^\mathrm{d} - \boldsymbol{r}\right\}, \tag{3.8}$$

for all $\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\eta}}\in\mathbb{H}$ and

$$\mathbf{C}(\boldsymbol{\chi},\boldsymbol{\psi}) \;:=\; \int_\Omega \mathbb{K}\nabla\boldsymbol{\chi} : \nabla\boldsymbol{\psi} \qquad \forall\,\boldsymbol{\chi},\boldsymbol{\psi}\in\mathbf{H}^1(\Omega). \tag{3.9}$$

In turn, $F_{\boldsymbol{\phi}}$ and $G_{\boldsymbol{w},\boldsymbol{\phi}}$ are linear functionals defined as

$$F_{\boldsymbol{\phi}}(\vec{\boldsymbol{\eta}}) \;:=\; \int_\Omega (\boldsymbol{\alpha}\cdot\boldsymbol{\phi})\boldsymbol{g}\cdot\{\boldsymbol{v} - \kappa_2\mathbf{div}\,\boldsymbol{\tau}\} \qquad \forall\,\vec{\boldsymbol{\eta}}\in\mathbb{H}, \tag{3.10}$$

and

$$G_{\boldsymbol{w},\boldsymbol{\phi}}(\boldsymbol{\psi}) \ := -\int_{\Omega} (\nabla\boldsymbol{\phi})\boldsymbol{w}\cdot\boldsymbol{\psi} \qquad \forall\,\boldsymbol{\psi}\in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,. \tag{3.11}$$

Remark 3.1. The model (2.1) describes the diffusive convection of a fluid driven by two solute components, like temperature and concentration, with different molecular diffusivities making opposing contributions to density gradient. Although we have developed the formulation (3.6) for this double-diffusive special case, our results in the rest of this section and Sections 4 and 5 can be adapted/extended to the more general case in which multiple scalars might simultaneously affect the fluid density, namely, multiple-diffusive natural convection phenomena or other related models, such as bio-thermal convection of oxytactic microorganisms [30].

Remark 3.2. As described in [19], an alternative way to formulate the transport equation (second equation of second row in (2.9)), in a mixed-primal form instead of the standard one given by (3.6b), may be based on the introduction of the normal derivative of $\boldsymbol{\varphi}$ as an additional unknown on the Dirichlet boundary and so the Dirichlet datum $\boldsymbol{\varphi}_{\mathrm{D}}$ would be weakly incorporated into the formulation. However, this would require not only to add a new variable into the system but also a pair of inf-sup compatible finite element spaces for guaranteeing the existence of a solution. For the sake of avoiding these facts and keeping the flexibility of the scheme as much as possible, we proceed differently than [19] by considering the standard primal formulation for the transport equation.

3.2. The fixed point approach

In this Section we describe a fixed point strategy that allows us to solve the coupled problem given by (3.6). We start by denoting $\mathbf{H} := \mathbf{H}^1_0(\Omega)\times\mathbf{H}^1(\Omega)$, and defining the operator $\mathcal{A}:\mathbf{H}\to\mathbb{H}$ by

$$\mathcal{A}(\boldsymbol{w},\boldsymbol{\phi}) = (\mathcal{A}_1(\boldsymbol{w},\boldsymbol{\phi}),\,\mathcal{A}_2(\boldsymbol{w},\boldsymbol{\phi}),\,\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi})) := \vec{\xi} \qquad \forall\,(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{H}$$

where $\vec{\xi} = (\boldsymbol{t},\boldsymbol{\sigma},\boldsymbol{u})\in\mathbb{H}$ is the unique solution of the augmented mixed formulation given by (3.6a), with $(\boldsymbol{w},\boldsymbol{\phi})$ instead $(\boldsymbol{u},\boldsymbol{\varphi})$, that is:

$$\mathbf{A}_{\boldsymbol{\phi}}(\vec{\xi},\vec{\eta}) + \mathbf{B}_{\boldsymbol{w}}(\vec{\xi},\vec{\eta}) \ = \ F_{\boldsymbol{\phi}}(\vec{\eta}) \qquad \forall\,\vec{\eta}\in\mathbb{H}\,, \tag{3.12}$$

where the bilinear forms $\mathbf{A}_{\boldsymbol{\phi}}$, $\mathbf{B}_{\boldsymbol{w}}$ and the functional $F_{\boldsymbol{\phi}}$ are defined exactly as in (3.7), (3.8), and (3.10), respectively. In addition, we also introduce the operator $\mathcal{B}:\mathbf{H}\to\mathbf{H}^1(\Omega)$ defined as

$$\mathcal{B}(\boldsymbol{w},\boldsymbol{\phi}) := \boldsymbol{\varphi} \qquad \forall\,(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{H},$$

where $\boldsymbol{\varphi}$ is the unique solution of the problem (3.6b), with $(\boldsymbol{w},\boldsymbol{\phi})$ instead $(\boldsymbol{u},\boldsymbol{\varphi})$, that is:

$$\mathbf{C}(\boldsymbol{\varphi},\boldsymbol{\psi}) \ = \ G_{\boldsymbol{w},\boldsymbol{\phi}}(\boldsymbol{\psi}) \qquad \forall\,\boldsymbol{\psi}\in\mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega), \tag{3.13}$$

where the bilinear form $\mathbf{C}$, and the functional $G_{\boldsymbol{w},\boldsymbol{\phi}}$ are defined by (3.9) and (3.11), respectively.

In this way, by introducing the operator $\mathscr{L}:\mathbf{H}\to\mathbf{H}$ as

$$\mathscr{L}(\boldsymbol{w},\boldsymbol{\phi}) := (\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi}),\,\mathcal{B}(\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi}),\boldsymbol{\phi})) \qquad \forall\,(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{H} \tag{3.14}$$

we readily realize that solving (3.6) is equivalent to seeking a fixed point of $\mathscr{L}$, that is: Find $(\boldsymbol{u},\boldsymbol{\varphi})\in\mathbf{H}$ such that

$$\mathscr{L}(\boldsymbol{u},\boldsymbol{\varphi}) = (\boldsymbol{u},\boldsymbol{\varphi})\,. \tag{3.15}$$

The following two sections establish the well-posedness of (3.15).

3.3. Well-posedness of the uncoupled problems

We begin by recalling the following lemmas which are useful to prove the ellipticity of the bilinear form $\mathbf{A}_{\boldsymbol{\phi}}+\mathbf{B}_{\boldsymbol{w}}$.

Lemma 3.1. *There exists $c_2(\Omega)>0$ such that*

$$c_2(\Omega)\|\boldsymbol{\tau}_0\|^2_{0,\Omega} \ \le \ \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{\tau}=\boldsymbol{\tau}_0+c\,\mathbb{I}\in\mathbb{H}(\mathbf{div};\Omega)\,,$$

with $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\Omega)$ and $c\in\mathrm{R}$.

Proof. See [9, Proposition 3.1]. □

Lemma 3.2. *There holds*

$$\frac{1}{2}|\boldsymbol{v}|^2_{1,\Omega} \le \|\mathbf{e}(\boldsymbol{v})\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)\,.$$

Proof. See [31, Theorem 10.1]. □

We now provide sufficient conditions under which the uncoupled problems (3.12) and (3.13) are indeed uniquely solvable.

Lemma 3.3. *Assume that* $\kappa_1 \in \big(0, 2\delta_3\big(2\nu_1 - \frac{\kappa_3\nu_2}{\delta_1}\big)\big)$, $\kappa_2 \in (0, 2\delta_2)$, $\kappa_3 \in \big(0, \frac{2\nu_1\delta_1}{\nu_2}\big)$ *with* $\delta_1 \in \big(0, \frac{1}{\nu_2}\big)$, $\delta_2 \in \big(0, \frac{2}{\gamma}\big)$, $\delta_3 \in (0,1)$. *Then, there exists* $r_0 > 0$, *depending only on* γ, ν_1, ν_2, κ_1, κ_2, κ_3, *and the continuous injection* $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ *(cf. (3.22)), such that for each* $r \in (0, r_0)$, *problem (3.12) has a unique solution* $\mathcal{A}(\boldsymbol{w},\boldsymbol{\phi}) := \vec{\xi} \in \mathbb{H}$, *for each* $(\boldsymbol{w},\boldsymbol{\phi}) \in \mathbf{H}$ *with* $\|\boldsymbol{w}\|_{1,\Omega} \le r$. *Moreover, there exists* $C_{\mathcal{A}}$, *independent of* $(\boldsymbol{w},\boldsymbol{\phi})$, *such that*

$$\|\mathcal{A}(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbb{H}} \;=\; \|\vec{\xi}\|_{\mathbb{H}} \;\le\; C_{\mathcal{A}}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\phi}\|_{0,\Omega} \quad \forall\,(\boldsymbol{w},\boldsymbol{\phi}) \in \mathbf{H}. \tag{3.16}$$

Proof. We begin by deriving the continuity of the bilinear forms $\mathbf{A}_{\boldsymbol{\phi}}$ and $\mathbf{B}_{\boldsymbol{w}}$ (sf. (3.7) and (3.8), respectively). Indeed, applying Cauchy-Schwarz's inequality, the assumptions (2.3), and the fact that $\|\mathbf{e}(\boldsymbol{v})\|_{0,\Omega} \le |\boldsymbol{v}|_{1,\Omega}$ and $\|\boldsymbol{\omega}(\boldsymbol{v})\|_{0,\Omega} \le |\boldsymbol{v}|_{1,\Omega}$ $\forall\,\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, we deduce that there exists a positive constant $C_{\mathbf{A}_{\boldsymbol{\phi}}}$, depending on $\kappa_1, \kappa_2, \kappa_3, \nu_2, \gamma$, such that

$$|\mathbf{A}_{\boldsymbol{\phi}}(\vec{\xi},\vec{\boldsymbol{\eta}})| \;\le\; C_{\mathbf{A}_{\boldsymbol{\phi}}}\|\vec{\xi}\|_{\mathbb{H}}\|\vec{\boldsymbol{\eta}}\|_{\mathbb{H}} \qquad \forall\,\vec{\xi},\vec{\boldsymbol{\eta}} \in \mathbb{H}, \tag{3.17}$$

where $C_{\mathbf{A}_{\boldsymbol{\phi}}} := 3\max\{2, \gamma + \kappa_1, \kappa_2 + \kappa_3, 2\nu_2, 1 + 2\nu_2\kappa_3, 1 + |1 - \gamma\kappa_2|\}$. In turn, by applying Hölder's inequality and the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, with constant $c_1(\Omega)$, we obtain that

$$|\mathbf{B}_{\boldsymbol{w}}(\vec{\xi},\vec{\boldsymbol{\eta}})| \;\le\; c_1(\Omega)(2+\kappa_3^2)^{1/2}\|\boldsymbol{w}\|_{1,\Omega}\,\|\vec{\xi}\|_{\mathbb{H}}\,\|\vec{\boldsymbol{\eta}}\|_{\mathbb{H}} \qquad \forall\,\vec{\xi},\vec{\boldsymbol{\eta}} \in \mathbb{H}. \tag{3.18}$$

Then, it follows from (3.17) and (3.18), that there exists a positive constant denote by $\|\mathbf{A}_{\boldsymbol{\varphi}} + \mathbf{B}_{\boldsymbol{w}}\|$, that depends on $\kappa_1, \kappa_2, \kappa_3, \nu_2, \gamma, c_1(\Omega)$ and $\|\boldsymbol{w}\|_{1,\Omega}$, such that

$$|(\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}})(\vec{\xi},\vec{\boldsymbol{\eta}})| \;\le\; \|\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}}\|\,\|\vec{\xi}\|_{\mathbb{H}}\,\|\vec{\boldsymbol{\eta}}\|_{\mathbb{H}} \qquad \forall\,\vec{\xi},\vec{\boldsymbol{\eta}} \in \mathbb{H}. \tag{3.19}$$

We now address the proof of the ellipticity for the bilinear form $\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}}$. To this end, we first derive this property for the bilinear form $\mathbf{A}_{\boldsymbol{\phi}}$. In fact, from (3.7), by applying (2.3) and the Cauchy-Schwarz inequality, together with Lemma 3.2, it follows that

$$\begin{aligned}
\mathbf{A}_{\boldsymbol{\phi}}(\vec{\xi},\vec{\xi}) = {} & 2\int_\Omega \nu(\boldsymbol{\phi})\boldsymbol{t} : \boldsymbol{t} \,-\, 2\kappa_3\int_\Omega \nu(\boldsymbol{\phi})\boldsymbol{t} : \boldsymbol{\sigma}^{\mathrm{d}} \,-\, \kappa_1\int_\Omega \boldsymbol{t} : \mathbf{e}(\boldsymbol{u}) \,+\, \kappa_3\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega}^2 \\
& -\, \kappa_2\gamma\int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\sigma} \,+\, \gamma\|\boldsymbol{u}\|_{0,\Omega}^2 \,+\, \kappa_2\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}^2 \,+\, \kappa_1\|\mathbf{e}(\boldsymbol{u})\|_{0,\Omega}^2 \\
\ge {} & 2\nu_1\|\boldsymbol{t}\|_{0,\Omega}^2 \,-\, 2\kappa_3\nu_2\|\boldsymbol{t}\|_{0,\Omega}\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega} \,-\, \kappa_1\|\boldsymbol{t}\|_{0,\Omega}\|\mathbf{e}(\boldsymbol{u})\|_{0,\Omega} \,+\, \kappa_3\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega}^2 \\
& -\, \kappa_2\gamma\|\boldsymbol{u}\|_{0,\Omega}\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega} \,+\, \gamma\|\boldsymbol{u}\|_{0,\Omega}^2 \,+\, \kappa_2\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}^2 \,+\, \kappa_1\|\mathbf{e}(\boldsymbol{u})\|_{0,\Omega}^2 \\
\ge {} & 2\nu_1\|\boldsymbol{t}\|_{0,\Omega}^2 \,-\, 2\kappa_3\nu_2\|\boldsymbol{t}\|_{0,\Omega}\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega} \,-\, \kappa_1\|\boldsymbol{t}\|_{0,\Omega}|\boldsymbol{u}|_{1,\Omega} \,+\, \kappa_3\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega}^2 \\
& -\, \kappa_2\gamma\|\boldsymbol{u}\|_{0,\Omega}\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega} \,+\, \gamma\|\boldsymbol{u}\|_{0,\Omega}^2 \,+\, \kappa_2\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}^2 \,+\, \frac{\kappa_1}{2}|\boldsymbol{u}|_{1,\Omega}^2
\end{aligned}$$

Next, employing the Young inequality and gathering similar terms, we obtain

$$\begin{aligned}
\mathbf{A}_{\boldsymbol{\phi}}(\vec{\xi},\vec{\xi}) \ge {} & \left(2\nu_1 - \frac{\kappa_3\nu_2}{\delta_1} - \frac{\kappa_1}{2\delta_3}\right)\|\boldsymbol{t}\|_{0,\Omega}^2 \,+\, \kappa_3\big(1 - \nu_2\delta_1\big)\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \kappa_2\left(1 - \frac{\gamma\delta_2}{2}\right)\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}^2 \\
& +\, \gamma\left(1 - \frac{\kappa_2}{2\delta_2}\right)\|\boldsymbol{u}\|_{0,\Omega}^2 \,+\, \frac{\kappa_1}{2}\big(1-\delta_3\big)|\boldsymbol{u}|_{1,\Omega}^2
\end{aligned}$$

and then, choosing κ_1, κ_2, κ_3, δ_1, δ_2 and δ_3 in the ranges specified of the statement of the present lemma, we deduce that there exists a positive constant $\alpha(\Omega)$, independent of $(\boldsymbol{w},\boldsymbol{\varphi})$, such that

$$\mathbf{A}_{\boldsymbol{\phi}}(\vec{\xi},\vec{\xi}) \;\ge\; \alpha(\Omega)\,\|\vec{\xi}\|_{\mathbb{H}}^2 \qquad \forall\,\vec{\xi} \in \mathbb{H}. \tag{3.20}$$

Next, by combining (3.18) and (3.20), we find that

$$(\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}})(\vec{\xi},\vec{\xi}) \;\ge\; \Big\{\alpha(\Omega) - c_1(\Omega)(2+\kappa_3^2)^{1/2}\|\boldsymbol{w}\|_{1,\Omega}\Big\}\|\vec{\xi}\|_{\mathbb{H}}^2 \qquad \forall\,\vec{\xi} \in \mathbb{H},$$

from which, we deduce that

$$(\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}})(\vec{\xi},\vec{\xi}) \;\ge\; \frac{\alpha(\Omega)}{2}\|\vec{\xi}\|_{\mathbb{H}}^2 \qquad \forall\,\vec{\xi} \in \mathbb{H}, \tag{3.21}$$

provided that $\|\boldsymbol{w}\|_{1,\Omega} \le r_0$, with

$$r_0 \;:=\; \frac{\alpha(\Omega)}{2c_1(\Omega)(2+\kappa_3^2)^{1/2}}, \tag{3.22}$$

which confirms the ellipticity of $\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}}$. On the other hand, by applying the Cauchy-Schwarz inequality, we deduce that $F_{\boldsymbol{\phi}} \in \mathbb{H}'$ (cf. (3.10)) with

$$\|F_{\boldsymbol{\phi}}\| \;\le\; \sqrt{2}\,(2+\kappa_2^2)^{1/2}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\phi}\|_{0,\Omega}. \tag{3.23}$$

Consequently, a straightforward application of the Lax-Milgram lemma implies that there exists a unique solution $\vec{\xi} \in \mathbb{H}$ of (3.12). Finally, from (3.21) and (3.23), and performing simple algebraic manipulations, we derive (3.16), with $C_{\mathcal{A}} := \dfrac{2\sqrt{2}\,(2+\kappa_2^2)^{1/2}}{\alpha(\Omega)} > 0$, independent of $(\boldsymbol{w},\boldsymbol{\phi})$. □

Remark 3.3. At this point, we remark that for computational purposes, the constant $\alpha(\Omega)$ defined in Lemma 3.3, can be maximized by choosing the parameters δ_1, δ_2, δ_3, κ_1, κ_2 and κ_3 at the middle points of their feasible ranges. Thus, adequate choices for these parameters are $\delta_1 := \frac{1}{2\nu_2}$, $\delta_2 := \frac{1}{\gamma}$ and $\delta_3 := \frac{1}{2}$, which establish that

$$\kappa_1 = \frac{\nu_1}{2}, \qquad \kappa_2 = \frac{1}{\gamma} \qquad \text{and} \qquad \kappa_3 = \frac{\nu_1}{2\nu_2^2}. \tag{3.24}$$

Remark 3.4. In order to establish the solvability of the problem (3.13), associated to the operator $\mathcal{B}$, we first point out that $\boldsymbol{\varphi}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$ can be continuously extended in the trace sense to $\mathbf{H}^{1/2}(\Gamma)$. Indeed, the set

$$\mathcal{H}(\boldsymbol{\varphi}_{\mathrm{D}}) = (\gamma_0|_{\Gamma_{\mathrm{D}}})^{-1}(\{\boldsymbol{\varphi}_{\mathrm{D}}\}) = \left\{\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega) : \quad \gamma_0(\boldsymbol{\varphi})|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}}\right\}$$

is a closed and convex since the usual trace operator $\gamma_0 : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ is linear and continuous. Then, from The Best Approximation Theorem [22, Theorem 7] there exists a unique $\widetilde{\boldsymbol{\varphi}} := E(\boldsymbol{\varphi}_{\mathrm{D}}) \in \mathbf{H}^1(\Omega)$, where E denotes the extension of $\boldsymbol{\varphi}_{\mathrm{D}}$, such that $\gamma_0(\widetilde{\boldsymbol{\varphi}})|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}}$ with

$$\|\widetilde{\boldsymbol{\varphi}}\|_{1,\Omega} = \inf_{\boldsymbol{\varphi}\in\mathcal{H}(\boldsymbol{\varphi}_{\mathrm{D}})} \|\boldsymbol{\varphi}\|_{1,\Omega} = \operatorname{dist}(\mathbf{0}, \mathcal{H}(\boldsymbol{\varphi}_{\mathrm{D}})) = \|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}. \tag{3.25}$$

In this way, the suitable extension of $\boldsymbol{\varphi}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$ is nothing but the corresponding one of $\gamma_0(\widetilde{\boldsymbol{\varphi}}) \in \mathbf{H}^{1/2}(\Gamma)$ to $\mathbf{H}^1(\Omega)$.

Lemma 3.4. *For each $(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{H}$, problem (3.13) has a unique solution $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$, with $\boldsymbol{\varphi}|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}}$. Moreover, there exists a constant $C_{\mathcal{B}} > 0$ independent of $(\boldsymbol{w}, \boldsymbol{\phi})$, such that*

$$\|\mathcal{B}(\boldsymbol{w}, \boldsymbol{\phi})\|_{1,\Omega} = \|\boldsymbol{\varphi}\|_{1,\Omega} \le C_{\mathcal{B}}\left\{\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{\phi}\|_{1,\Omega} + \|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\}. \tag{3.26}$$

Proof. We begin by nothing, according to the aforementioned in Remark 3.4, that given $\boldsymbol{\varphi}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$ there exists $\boldsymbol{\varphi}_1 \in \mathbf{H}^1(\Omega)$ such that

$$\boldsymbol{\varphi}_1|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}} \qquad \text{and} \qquad \|\boldsymbol{\varphi}_1\|_{1,\Omega} = \|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}. \tag{3.27}$$

Then, we consider the auxiliary linear problem: Find $\boldsymbol{\varphi}_0 \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$\mathbf{C}(\boldsymbol{\varphi}_0, \boldsymbol{\psi}) = G_{\boldsymbol{w},\boldsymbol{\phi}}(\boldsymbol{\psi}) - \mathbf{C}(\boldsymbol{\varphi}_1, \boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) \tag{3.28}$$

where $\mathbf{C}$ and $G_{\boldsymbol{w},\boldsymbol{\phi}}$ are defined in (3.9) and (3.11), respectively. In addition, from (3.9) and the Cauchy-Schwarz's inequality, we deduce that

$$|\mathbf{C}(\boldsymbol{\chi}, \boldsymbol{\psi})| \le \|\mathbb{K}\|_{\infty,\Omega}\|\boldsymbol{\chi}\|_{1,\Omega}\|\boldsymbol{\psi}\|_{1,\Omega} \qquad \forall\, \boldsymbol{\chi}, \boldsymbol{\psi} \in \mathbf{H}^1(\Omega), \tag{3.29}$$

which, in particular, confirms the boundedness of $\mathbf{C}$. Then, from (3.9), using (2.2) and the Poincaré inequality (3.1), we get

$$\mathbf{C}(\boldsymbol{\psi}, \boldsymbol{\psi}) \ge \widetilde{\alpha}\,\|\boldsymbol{\psi}\|^2_{1,\Omega} \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega), \tag{3.30}$$

which proves that $\mathbf{C}$ is $\mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$-elliptic with constant

$$\widetilde{\alpha} := k_0\, c_{\mathrm{gp}}^{-1}. \tag{3.31}$$

Next, applying the Cauchy-Schwarz's inequality, the boundedness of the continuous injection $\boldsymbol{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, with constant $c_1(\Omega) := \|\boldsymbol{i}\|^2$, (3.29), and the second identity in (3.27), we easily deduce that

$$|G_{\boldsymbol{w},\boldsymbol{\phi}}(\boldsymbol{\psi}) - \mathbf{C}(\boldsymbol{\varphi}_1, \boldsymbol{\psi})| \le \left\{c_1(\Omega)\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{\phi}\|_{1,\Omega} + \|\mathbb{K}\|_{\infty,\Omega}\|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\}\|\boldsymbol{\psi}\|_{1,\Omega} \quad \forall\, \boldsymbol{\psi} \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega), \tag{3.32}$$

which establishes the boundedness of the right-hand side of (3.28). Consequently, a direct application of the Lax-Milgram lemma implies that there exists a unique $\boldsymbol{\varphi}_0 \in \mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ that satisfies (3.28) with

$$\|\boldsymbol{\varphi}_0\|_{1,\Omega} \le \frac{1}{\widetilde{\alpha}}\left\{c_1(\Omega)\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{\phi}\|_{1,\Omega} + \|\mathbb{K}\|_{\infty,\Omega}\|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\}.$$

On the other hand, we now set $\boldsymbol{\varphi} := \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_1$, which satisfies that $\boldsymbol{\varphi}|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_0|_{\Gamma_{\mathrm{D}}} + \boldsymbol{\varphi}_1|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}}$. In addition, it is easy to check that $\boldsymbol{\varphi}$ is a solution of problem (3.13), where the uniqueness comes from (3.30). Finally, $\boldsymbol{\varphi}$ verifies the estimate (3.26), with constant $C_{\mathcal{B}} := \max\{\widetilde{\alpha}^{-1}c_1(\Omega), \widetilde{\alpha}^{-1}\|\mathbb{K}\|_{\infty,\Omega} + 1\}$. □

We end this section by introducing a further regularity hypotheses on the problem defining $\mathcal{A}$, which will be employed to derive a Lipschitz-continuity property for the operator $\mathscr{L}$. More precisely, we assume that for each $(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{H}$ with $\|\boldsymbol{w}\|_{1,\Omega} + \|\boldsymbol{\phi}\|_{1,\Omega} \le r$, $r > 0$ given, there holds $(\boldsymbol{r}, \boldsymbol{\tau}, \boldsymbol{v}) = \mathcal{A}(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{H}^{\varepsilon}(\Omega) \times \mathbb{H}_0(\mathbf{div}; \Omega) \cap \mathbb{H}^{\varepsilon}(\Omega) \times \mathbf{H}^{1+\varepsilon}(\Omega)$, for some $\varepsilon \in (0, 1)$ (when $n = 2$) or $\varepsilon \in \left[\frac{1}{2}, 1\right)$ (when $n = 3$), with

$$\|\boldsymbol{r}\|_{\varepsilon,\Omega} + \|\boldsymbol{\tau}\|_{\varepsilon,\Omega} + \|\boldsymbol{v}\|_{\varepsilon,\Omega} \le \widehat{C}(r)\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\phi}\|_{0,\Omega}, \tag{3.33}$$

where $\widehat{C}(r)$ is a positive constant independent of $(\boldsymbol{w}, \boldsymbol{\phi})$ but depending on the upper bound r of its norm. The reason of the estipulated ranges for ε will be clarified in the forthcoming analysis (specifically in the proofs of Lemmas 3.6 and 3.8 below). Also, we pay attention to the fact the while the estimate (3.33) will be employed only to bound $\|\boldsymbol{r}\|_{\varepsilon,\Omega}$, we have stated it including the terms $\|\boldsymbol{\tau}\|_{\varepsilon,\Omega}$ and $\|\boldsymbol{v}\|_{1+\varepsilon,\Omega}$ as well, since due to the first and fourth equations of (2.9), the regularities of $\boldsymbol{r}, \boldsymbol{\tau}$ and $\boldsymbol{v}$ will most likely be connected.

3.4. Solvability analysis of the fixed point equation

We begin by emphasizing that the well-posedness of the uncoupled problems (3.12) and (3.13) confirms that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathscr{L}$ (cf. (3.14)) are well defined, and hence now we can address the solvability analysis of the fixed point problem presented in (3.15). To this end, we will verify below the hypotheses of the Banach fixed-point theorem.

Lemma 3.5. *Given $r \in (0, r_0)$, with r_0 given by (3.22), we let*

$$W_r := \left\{(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{H} : \quad \|(\boldsymbol{w}, \boldsymbol{\phi})\|_{\mathbf{H}} \le r\right\},$$

and assume that the data satisfy

$$c(r)\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}| + C_{\mathcal{B}}\,\|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \le r \tag{3.34}$$

where $c(r) := r C_{\mathcal{A}}(1 + r C_{\mathcal{B}})$, with $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ given by (3.16) and (3.26), respectively. Then, there holds $\mathscr{L}(W_r) \subseteq W_r$.

Proof. It follows similar as in [19, Lemma 3.5] □

Lemma 3.6. *Let $r \in (0, r_0)$ with r_0 given by (3.22). Then, there exists a constant $L_{\mathcal{A}} > 0$, depending on the stabilization parameters κ_2, κ_3, and the constants $L_\nu, c_1(\Omega), \alpha(\Omega), C_\varepsilon$ (cf. (2.3), (3.3), (3.20) and (3.41), respectively), such that for all $(\boldsymbol{w}, \boldsymbol{\phi}), (\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\phi}}) \in \mathbf{H}$, with $\|\boldsymbol{w}\|_{1,\Omega}, \|\widetilde{\boldsymbol{w}}\|_{1,\Omega} \le r$, there holds*

$$\begin{aligned}\|\mathcal{A}(\boldsymbol{w}, \boldsymbol{\phi}) - \mathcal{A}(\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\phi}})\|_{\mathbb{H}} \le L_{\mathcal{A}}\Big\{&\|\mathcal{A}_3(\boldsymbol{w}, \boldsymbol{\phi})\|_{1,\Omega}\,\|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} \\ &+ \|\mathcal{A}_1(\boldsymbol{w}, \boldsymbol{\phi})\|_{\varepsilon,\Omega}\,\|\boldsymbol{\phi} - \widetilde{\boldsymbol{\phi}}\|_{0,n/\varepsilon,\Omega} + |\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\phi} - \widetilde{\boldsymbol{\phi}}\|_{1,\Omega}\Big\}.\end{aligned} \tag{3.35}$$

Proof. Given $(\boldsymbol{w}, \boldsymbol{\phi}), (\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\phi}})$ as stated, we let $\vec{\boldsymbol{\xi}} = (\boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{u}) = \mathcal{A}(\boldsymbol{w}, \boldsymbol{\phi})$ and $\vec{\boldsymbol{\zeta}} = (\widetilde{\boldsymbol{t}}, \widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}) = \mathcal{A}(\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\phi}})$, which according to the definition of operator $\mathcal{A}$ (cf. (3.12)), means that:

$$\mathbf{A}_{\boldsymbol{\phi}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\eta}}) + \mathbf{B}_{\boldsymbol{w}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\eta}}) = F_{\boldsymbol{\phi}}(\vec{\boldsymbol{\eta}}) \qquad \text{and} \qquad \mathbf{A}_{\widetilde{\boldsymbol{\phi}}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}) + \mathbf{B}_{\widetilde{\boldsymbol{w}}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}) = F_{\widetilde{\boldsymbol{\phi}}}(\vec{\boldsymbol{\eta}}) \qquad \forall\, \vec{\boldsymbol{\eta}} \in \mathbb{H}.$$

Then, subtracting both identities, replacing $\vec{\boldsymbol{\zeta}} = \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\xi}} + \vec{\boldsymbol{\xi}}$, and using the bilinearity of $\mathbf{A}_{\boldsymbol{\phi}} + \mathbf{B}_{\boldsymbol{w}}$ for any $\boldsymbol{\phi}$ and $\boldsymbol{w}$, it follows from (3.12) that:

$$(\mathbf{A}_{\widetilde{\boldsymbol{\phi}}} + \mathbf{B}_{\widetilde{\boldsymbol{w}}})(\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}) = (F_{\boldsymbol{\phi}} - F_{\widetilde{\boldsymbol{\phi}}})(\vec{\boldsymbol{\eta}}) + (\mathbf{A}_{\widetilde{\boldsymbol{\phi}}} - \mathbf{A}_{\boldsymbol{\phi}})(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\eta}}) + \mathbf{B}_{\widetilde{\boldsymbol{w}} - \boldsymbol{w}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\eta}}) \qquad \forall\, \vec{\boldsymbol{\eta}} \in \mathbb{H}. \tag{3.36}$$

Moreover, applying the ellipticity of $\mathbf{A}_{\widetilde{\boldsymbol{\phi}}} + \mathbf{B}_{\widetilde{\boldsymbol{w}}}$ (cf. (3.21)), and then employing (3.36) with $\vec{\boldsymbol{\eta}} := \vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}$, we find that

$$\begin{aligned}\frac{\alpha(\Omega)}{2}\|\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}\|_{\mathbb{H}}^2 &\le (\mathbf{A}_{\widetilde{\boldsymbol{\phi}}} + \mathbf{B}_{\widetilde{\boldsymbol{w}}})(\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}) \\ &= (F_{\boldsymbol{\phi}} - F_{\widetilde{\boldsymbol{\phi}}})(\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}) + (\mathbf{A}_{\widetilde{\boldsymbol{\phi}}} - \mathbf{A}_{\boldsymbol{\phi}})(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}) + \mathbf{B}_{\widetilde{\boldsymbol{w}} - \boldsymbol{w}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}).\end{aligned} \tag{3.37}$$

Then, for the first and third terms on the right-hand side of (3.37), we employ the Cauchy-Schwarz and Hölder inequalities, together with the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, similar as in (3.23) and (3.18), in order to obtain that

$$\begin{aligned}|(F_{\boldsymbol{\phi}} - F_{\widetilde{\boldsymbol{\phi}}})(\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}})| &= \left|\int_\Omega (\boldsymbol{\alpha}(\boldsymbol{\phi} - \widetilde{\boldsymbol{\phi}}))\boldsymbol{g} \cdot \left\{(\boldsymbol{u} - \widetilde{\boldsymbol{u}}) - \kappa_2\,\mathbf{div}(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\right\}\right| \\ &\le \sqrt{2}\,(2 + \kappa_2^2)^{1/2}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\phi} - \widetilde{\boldsymbol{\phi}}\|_{1,\Omega}\,\|\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}\|_{\mathbb{H}}\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}|\mathbf{B}_{\boldsymbol{w} - \widetilde{\boldsymbol{w}}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}})| &= \left|\int_\Omega (\boldsymbol{u} \otimes (\widetilde{\boldsymbol{w}} - \boldsymbol{w}))^{\mathrm{d}} : \left\{\kappa_3(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})^{\mathrm{d}} - (\boldsymbol{t} - \widetilde{\boldsymbol{t}})\right\}\right| \\ &\le c_1(\Omega)\,(2 + \kappa_3^2)^{1/2}\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega}\,\|\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}\|_{\mathbb{H}}\end{aligned} \tag{3.39}$$

On the other hand, for the second term in the right-hand side of (3.37), we apply the Lipschitz continuity property for ν given in (2.3), the Cauchy-Schwarz and Hölder inequalities, and the definition of $\mathbf{A}_{\boldsymbol{\varphi}}$ (cf. (3.7)), to obtain that

$$\begin{aligned}|(\mathbf{A}_{\widetilde{\boldsymbol{\phi}}} - \mathbf{A}_{\boldsymbol{\phi}})(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}})| &= \left|\int_\Omega 2(\nu(\boldsymbol{\phi}) - \nu(\widetilde{\boldsymbol{\phi}}))\boldsymbol{t} : \{(\boldsymbol{t} - \widetilde{\boldsymbol{t}}) - \kappa_3(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})^{\mathrm{d}}\}\right| \\ &\le 2L_\nu(2 + \kappa_3^2)^{1/2}\|\boldsymbol{\phi} - \widetilde{\boldsymbol{\phi}}\|_{0,2q,\Omega}\|\boldsymbol{t}\|_{0,2p,\Omega}\,\|\vec{\boldsymbol{\xi}} - \vec{\boldsymbol{\zeta}}\|_{\mathbb{H}},\end{aligned} \tag{3.40}$$

with $p, q \in (1, +\infty)$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We now proceed as in [7, Lemma 3.9]. In fact, given the further ε-regularity assumed in (3.33), we recall that the Sobolev embedding theorem (see e.g. [2, Theorem 4.12]) establishes the continuous injection $i_\varepsilon : \mathrm{H}^\varepsilon(\Omega) \to \mathrm{L}^{2p}(\Omega)$ with boundedness constant C_ε, where

$$2p = \begin{cases} \dfrac{2}{1 - \varepsilon} & \text{if } n = 2 \\[2ex] \dfrac{6}{3 - 2\varepsilon} & \text{if } n = 3 \end{cases}$$

and $2q = \frac{n}{\varepsilon}$, and therefore, there holds

$$\|\boldsymbol{t}\|_{0,2p,\Omega} \leq C_\varepsilon \|\boldsymbol{t}\|_{\varepsilon,\Omega} \qquad \forall\, \boldsymbol{t} \in \mathbb{H}^\varepsilon(\Omega). \tag{3.41}$$

In this way, denoting

$$L_{\mathcal{A}} := \frac{2}{\alpha(\Omega)} \max\left\{ c_1(\Omega)(2+\kappa_3^2)^{1/2},\ 2\,L_\nu(2+\kappa_3^2)^{1/2} C_\varepsilon,\ \sqrt{2}(2+\kappa_2^2)^{1/2} \right\}$$

from the inequalities (3.37), (3.38), (3.39), (3.40), and recalling that $\boldsymbol{t} = \mathcal{A}_1(\boldsymbol{w},\boldsymbol{\phi})$ and $\boldsymbol{u} = \mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi})$, yields (3.35) and concludes the proof. □

Lemma 3.7. *There exists $L_{\mathcal{B}} > 0$, depending on $\widetilde{\alpha}$ and $c_1(\Omega)$ (cf. (3.30) and (3.32), respectively) such that for all $(\boldsymbol{w},\boldsymbol{\phi}), (\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}}) \in W_r$ (cf. Lemma 3.5) there holds*

$$\|\mathcal{B}(\boldsymbol{w},\boldsymbol{\phi}) - \mathcal{B}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{1,\Omega} \leq L_{\mathcal{B}}\Big\{ \|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega} + \|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega} \Big\}. \tag{3.42}$$

Proof. Given $(\boldsymbol{w},\boldsymbol{\phi}), (\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}}) \in W_r$, we let $\boldsymbol{\varphi}, \widetilde{\boldsymbol{\varphi}} \in \mathbf{H}^1(\Omega)$ be the corresponding solutions of (3.13), that is $\boldsymbol{\varphi} := \mathcal{B}(\boldsymbol{w},\boldsymbol{\phi})$ and $\widetilde{\boldsymbol{\varphi}} := \mathcal{B}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})$. Then, since $\boldsymbol{\varphi}|_{\Gamma_{\mathrm{D}}} = \widetilde{\boldsymbol{\varphi}}|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}}$, we realize that $\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}$ belongs to $\mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$. In this way, applying the ellipticity of $\mathbf{C}$ (cf. (3.30)), using (3.13) and (3.11), adding and subtracting $\widetilde{\boldsymbol{\phi}}$, and employing the Hölder inequality, the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, and the definition of W_r (cf. Lemma 3.5), we readily deduce that

$$\begin{aligned}
\widetilde{\alpha}\, \|\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}\|^2_{1,\Omega} &\leq \mathbf{C}(\boldsymbol{\varphi},\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}) - \mathbf{C}(\widetilde{\boldsymbol{\varphi}},\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}) \\
&= G_{\boldsymbol{w},\boldsymbol{\phi}}(\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}) - G_{\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}}}(\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}) \\
&= -\int_\Omega \Big\{ \big(\nabla(\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}})\big)\boldsymbol{w} + (\nabla\widetilde{\boldsymbol{\phi}})(\boldsymbol{w}-\widetilde{\boldsymbol{w}}) \Big\} \cdot (\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}) \\
&\leq c_1(\Omega)\Big\{ \|\boldsymbol{w}\|_{1,\Omega}|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}|_{1,\Omega} + |\widetilde{\boldsymbol{\phi}}|_{1,\Omega}\|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega} \Big\}\|\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}\|_{1,\Omega} \\
&\leq r c_1(\Omega)\Big\{ \|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega} + \|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega} \Big\}\|\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}\|_{1,\Omega}
\end{aligned} \tag{3.43}$$

which give (3.42) with $L_{\mathcal{B}} := \frac{r c_1(\Omega)}{\widetilde{\alpha}}$. □

Lemma 3.8. *Let r and W_r as in Lemma 3.5. Then, there exists a positive constant $L_{\mathtt{Lip}}$, depending on r, $|\boldsymbol{\alpha}|$, $|\boldsymbol{g}|$, and the constants $C_{\mathcal{A}}$, $L_{\mathcal{A}}$, and $L_{\mathcal{B}}$ ((3.16), (3.35), and (3.42), respectively), such that for all $(\boldsymbol{w},\boldsymbol{\phi}), (\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}}) \in W_r$ there holds*

$$\|\mathscr{L}(\boldsymbol{w},\boldsymbol{\phi}) - \mathscr{L}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{\mathbf{H}} \leq L_{\mathtt{Lip}}\, \|(\boldsymbol{w},\boldsymbol{\phi}) - (\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{\mathbf{H}}. \tag{3.44}$$

Proof. Given $r \in (0, r_0)$ and $(\boldsymbol{w},\boldsymbol{\phi}), (\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}}) \in W_r$, we first observe, according to the definition of $\mathscr{L}$ (cf. (3.14)), and the Lipschitz-continuity of $\mathcal{B}$ (cf. (3.42)), that

$$\begin{aligned}
\|\mathscr{L}(\boldsymbol{w},\boldsymbol{\phi}) - \mathscr{L}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{\mathbf{H}} &= \|\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi}) - \mathcal{A}_3(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{1,\Omega} + \|\mathcal{B}(\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi}),\boldsymbol{\phi}) - \mathcal{B}(\mathcal{A}_3(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}}),\widetilde{\boldsymbol{\phi}})\|_{1,\Omega} \\
&\leq (1+L_{\mathcal{B}})\, \|\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi}) - \mathcal{A}_3(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{1,\Omega} + L_{\mathcal{B}}\, \|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega}
\end{aligned}$$

from which, employing the Lipschitz-continuity of $\mathcal{A}$ (cf. (3.6)), yields

$$\begin{aligned}
\|\mathscr{L}(\boldsymbol{w},\boldsymbol{\phi}) - \mathscr{L}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{\mathbf{H}} \leq\ & (1+L_{\mathcal{B}})L_{\mathcal{A}}\Big\{ \|\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi})\|_{1,\Omega}\, \|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega} \\
&+ \|\mathcal{A}_1(\boldsymbol{w},\boldsymbol{\phi})\|_{\varepsilon,\Omega}\, \|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{0,n/\varepsilon,\Omega} + |\boldsymbol{\alpha}|\,|\boldsymbol{g}|\, \|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega} \Big\} + L_{\mathcal{B}}\, \|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega}
\end{aligned} \tag{3.45}$$

Then, applying the bound (3.16) to estimate the term $\|\mathcal{A}_3(\boldsymbol{w},\boldsymbol{\phi})\|_{1,\Omega}$, employing the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^{n/\varepsilon}(\Omega)$ with boundedness constant $\widetilde{C}_\varepsilon$, using the estimate (3.33) to estimate the term $\|\mathcal{A}_1(\boldsymbol{w},\boldsymbol{\phi})\|_{\varepsilon,\Omega}$, noting that $\|\boldsymbol{\phi}\|_{1,\Omega} \leq r$, and performing some algebraic manipulations, we get from (3.45) that

$$\begin{aligned}
\|\mathscr{L}(\boldsymbol{w},\boldsymbol{\phi}) - \mathscr{L}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{\mathbf{H}} &\leq (1+L_{\mathcal{B}})L_{\mathcal{A}}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\Big\{ rC_{\mathcal{A}}\|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega} + (r\widetilde{C}_\varepsilon \widehat{C}(r)+1)\|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega} \Big\} \\
&\quad + L_{\mathcal{B}}\|\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}}\|_{1,\Omega} \\
&\leq L_{\mathtt{Lip}}\|(\boldsymbol{w},\boldsymbol{\phi}) - (\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\phi}})\|_{\mathbf{H}}
\end{aligned}$$

In this way, (3.44) follows from the foregoing inequality by defining

$$L_{\mathtt{Lip}} := rC_{\mathcal{A}}(1+L_{\mathcal{B}})L_{\mathcal{A}}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}| + (1+L_{\mathcal{B}})L_{\mathcal{A}}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|(r\widetilde{C}_\varepsilon\widehat{C}(r)+1) + L_{\mathcal{B}} \tag{3.46}$$

in order to complete the proof. □

Theorem 3.9. *Suppose that the parameters κ_1, κ_2 and κ_3 satisfy the conditions required by Lemma 3.3. Let r and W_r as in Lemma 3.5, and assume that the data satisfy (3.34) and allow to have*

$$L_{\mathtt{Lip}} < 1, \tag{3.47}$$

with $L_{\mathtt{Lip}}$ defined in (3.46). Then, problem (3.6) has a unique solution $(\vec{\xi}, \boldsymbol{\varphi}) \in \mathbb{H} \times \mathbf{H}^1(\Omega)$ such that $\boldsymbol{\varphi}|_{\Gamma_\mathrm{D}} = \boldsymbol{\varphi}_\mathrm{D}$, with $(\boldsymbol{u}, \boldsymbol{\varphi}) \in W_r$ (cf. Lemma 3.5), and there holds

$$\|\vec{\xi}\|_{\mathbb{H}} \le C_{\mathcal{A}}\, r\, |\boldsymbol{\alpha}|\, |\boldsymbol{g}|, \tag{3.48}$$

and

$$\|\boldsymbol{\varphi}\|_{1,\Omega} \le C_{\mathcal{B}} \left\{ r\, \|\boldsymbol{u}\|_{1,\Omega} + \|\boldsymbol{\varphi}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \right\}.$$

Proof. It follows as a combination of Lemmas 3.5 and 3.8, the assumption (3.47), the Banach fixed-point theorem, and the a priori estimates (3.16) and (3.26). We omit further details. □

4. Galerkin scheme

In this section we introduce and analyze the Galerkin scheme of the semi-augmented mixed-primal problem (3.6). To this end, we now let $\mathcal{T}_h$ be a regular triangulation of Ω by triangles K (in R^2) or tetrahedra K (in R^3) of diameter h_K, and define the mesh size $h := \max\{h_K : K \in \mathcal{T}_h\}$. In addition, given an integer $k \ge 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_k(K)$ be the space of polynomial functions on K of degree $\le k$, and define the corresponding local Raviart-Thomas space of order k as

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\,\boldsymbol{x},$$

where, according to the notations described in Section 1.1, $\mathbf{P}_k(K) = [\mathrm{P}_k(K)]^n$, and $\boldsymbol{x}$ is the generic vector in R^n. Then, we consider piecewise polynomials of degree $\le k$ for approximating entries of the strain rate $\boldsymbol{t}$, the global Raviart-Thomas space of order k to approximate rows of the pseudostress $\boldsymbol{\sigma}$, and the Lagrange space given by the continuous piecewise polynomial vectors of degree $\le k+1$ for the velocity $\boldsymbol{u}$, respectively, that is

$$\mathbb{H}_h^{\boldsymbol{t}} := \left\{ \boldsymbol{r}_h \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) : \quad \boldsymbol{r}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{4.1}$$

$$\mathbf{H}_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathrm{R}^n, \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{4.2}$$

$$\mathbf{H}_h^{\boldsymbol{u}} := \left\{ \boldsymbol{v}_h \in \mathbf{C}(\overline{\Omega}) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h, \quad \boldsymbol{v}_h = \mathbf{0} \quad \text{on} \quad \Gamma \right\}. \tag{4.3}$$

For the unknown $\boldsymbol{\varphi}$ containing the temperature and the concentration into its coordinates, we let $\mathbf{H}_h^{\boldsymbol{\varphi}} \subset \mathbf{H}^1(\Omega)$ denote the Lagrange space of degree $\le k+1$ with respect to $\mathcal{T}_h$ (similar to $\mathbf{H}_h^{\boldsymbol{u}}$), and set

$$\mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_\mathrm{D}} := \left\{ \boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}} : \quad \boldsymbol{\psi}_h|_{\Gamma_\mathrm{D}} = \mathbf{0} \right\} \tag{4.4}$$

to be the analogous space with homogeneous Dirichlet boundary conditions. We define

$$\boldsymbol{\varphi}_{\mathrm{D},h} := \mathcal{I}_h^{SZ}(E(\boldsymbol{\varphi}_\mathrm{D}))\Big|_{\Gamma_\mathrm{D}} \tag{4.5}$$

to be the approximate Dirichlet boundary data, where $\mathcal{I}_h^{SZ} : \mathbf{H}^1(\Omega) \to \mathbf{H}_h^{\boldsymbol{\varphi}}$ denotes the Scott-Zhang interpolant operator of degree $k+1$, which satisfies the following stability and approximation properties, respectively, (see [23, Lemma 1.130]).

Lemma 4.1. *Let p and ℓ satisfy $1 \le p < +\infty$ and $\ell \ge 1$ if $p = 1$, and $1/p < \ell$ otherwise. Then, there exists a positive constant c, independent of h, such that the following properties hold:*

(i) *For all $0 \le m \le min\{1, \ell\}$,*

$$\forall h, \ \forall v \in W^{\ell,p}(\Omega), \qquad \|\mathcal{I}_h^{SZ}(v)\|_{m,p,\Omega} \le c\, \|v\|_{\ell,p,\Omega}. \tag{4.6}$$

(ii) *Provided $\ell \le k+1$, for all $0 \le m \le \ell$,*

$$\forall h, \ \forall K \in \mathcal{T}_h, \ \forall v \in W^{\ell,p}(\Delta_K), \qquad \|v - \mathcal{I}_h^{SZ}(v)\|_{m,p,K} \le c\, h_K^{\ell-m}\, |v|_{\ell,p,\Delta_K} \tag{4.7}$$

where Δ_K denotes the set of elements in $\mathcal{T}_h$, sharing at least one vertex with K.

Hence, $\boldsymbol{\varphi}_{\mathrm{D},h}$ belongs to the discrete trace space on Γ_D given by

$$\mathbf{H}_h^{1/2}(\Gamma_\mathrm{D}) := \left\{ \boldsymbol{\psi}_{\mathrm{D},h} \in \mathbf{C}(\Gamma_\mathrm{D}) : \quad \boldsymbol{\psi}_{\mathrm{D},h}|_e \in \mathbf{P}_{k+1}(e) \quad \forall\, e \in \mathcal{E}_{\Gamma_\mathrm{D}} \right\},$$

where $\mathcal{E}_{\Gamma_\mathrm{D}}$ stands for the set of edges/faces on Γ_D.

In this way, defining $\mathbb{H}_h := \mathbb{H}_h^{\boldsymbol{t}} \times \mathbf{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ and denoting $\vec{\xi}_h := (\boldsymbol{t}_h, \boldsymbol{\sigma}_h, \boldsymbol{u}_h)$, the underlying Galerkin scheme given by the discrete counterpart of (3.6), reads: Find $(\vec{\xi}_h, \boldsymbol{\varphi}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{\varphi}}$, with $\boldsymbol{\varphi}_h\Big|_{\Gamma_\mathrm{D}} = \boldsymbol{\varphi}_{\mathrm{D},h}$, such that

$$\mathbf{A}_{\varphi_h}(\vec{\xi}_h,\vec{\eta}_h)+\mathbf{B}_{u_h}(\vec{\xi}_h,\vec{\eta}_h)=F_{\varphi_h}(\vec{\eta}_h) \qquad \forall\,\vec{\eta}_h\in\mathbb{H}_h, \tag{4.8a}$$

$$\mathbf{C}(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h)=G_{u_h,\varphi_h}(\boldsymbol{\psi}_h) \qquad \forall\,\boldsymbol{\psi}_h\in\mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_{\mathrm{D}}}. \tag{4.8b}$$

Throughout the rest of this section we adopt the discrete analogue of the fixed point strategy introduced in Section 3.2. Indeed, denoting $\mathbf{H}_h := \mathbf{H}^{\boldsymbol{u}}_h\times\mathbf{H}^{\boldsymbol{\varphi}}_h$, we define the operator $\mathcal{A}_h:\mathbf{H}_h\to\mathbb{H}_h$ by

$$\mathcal{A}_h(\boldsymbol{w}_h,\boldsymbol{\phi}_h) \;=\; (\mathcal{A}_{1,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h),\mathcal{A}_{2,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h),\mathcal{A}_{3,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h)) \;:=\; \vec{\xi}_h \qquad \forall\,(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\in\mathbf{H}_h$$

where $\vec{\xi}_h=(\boldsymbol{t}_h,\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\in\mathbb{H}_h$ is the unique solution of the discrete problem (4.8a) with $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)$ instead pf $(\boldsymbol{u}_h,\boldsymbol{\varphi}_h)$, that is

$$\mathbf{A}_{\phi_h}(\vec{\xi}_h,\vec{\eta}_h)+\mathbf{B}_{w_h}(\vec{\xi}_h,\vec{\eta}_h) \;=\; F_{\phi_h}(\vec{\eta}_h) \qquad \forall\,\vec{\eta}_h\in\mathbb{H}, \tag{4.9}$$

where the bilinear forms $\mathbf{A}_{\phi_h}$, $\mathbf{B}_{w_h}$ and the functional F_{ϕ_h} are those corresponding to (3.7), (3.8), and (3.10), respectively, with $\boldsymbol{w}=\boldsymbol{w}_h$ and $\boldsymbol{\phi}=\boldsymbol{\phi}_h$.

In addition, we introduce the operator $\mathcal{B}_h:\mathbf{H}_h\to\mathbf{H}^{\boldsymbol{\varphi}}_h$ defined as

$$\mathcal{B}_h(\boldsymbol{w}_h,\boldsymbol{\phi}_h) \;:=\; \boldsymbol{\varphi}_h \qquad \forall\,(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\in\mathbf{H}_h\,,$$

where $\boldsymbol{\varphi}_h$ is the unique solution of the discrete problem (4.8b) with $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)$ instead pf $(\boldsymbol{u}_h,\boldsymbol{\varphi}_h)$, that is

$$\mathbf{C}(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h) \;=\; G_{w_h,\phi_h}(\boldsymbol{\psi}_h) \qquad \forall\,\boldsymbol{\psi}_h\in\mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_{\mathrm{D}}}\,, \tag{4.10}$$

where the bilinear form $\mathbf{C}$, and the functional G_{w_h,ϕ_h} are defined as in (3.9) and (3.11), respectively, with $\boldsymbol{w}=\boldsymbol{w}_h$ and $\boldsymbol{\phi}=\boldsymbol{\phi}_h$.

Therefore, by introducing the operator $\mathscr{L}_h:\mathbf{H}_h\to\mathbf{H}_h$ as

$$\mathscr{L}_h(\boldsymbol{w}_h,\boldsymbol{\phi}_h) \;:=\; (\mathcal{A}_{3,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h),\,\mathcal{B}_h(\mathcal{A}_{3,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h),\boldsymbol{\phi}_h)) \qquad \forall\,(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\in\mathbf{H}_h$$

we realize that solving (4.8) is equivalent to seeking a fixed point of $\mathscr{L}_h$, that is: Find $(\boldsymbol{u}_h,\boldsymbol{\varphi}_h)\in\mathbf{H}_h$ such that

$$\mathscr{L}_h(\boldsymbol{u}_h,\boldsymbol{\varphi}_h) \;=\; (\boldsymbol{u}_h,\boldsymbol{\varphi}_h)\,. \tag{4.11}$$

Certainly, all the above makes sense if we guarantee that the discrete problems (4.9) and (4.10) are well-posed. This is precisely the purpose of the next section.

4.1. Well-posedness of the uncoupled problems

In this section, we establish the well-posedness of both (4.9) and (4.10), thus confirming that the operators $\mathcal{A}_h$, $\mathcal{B}_h$, and hence $\mathscr{L}_h$, are well-defined. We begin with the corresponding result for $\mathcal{A}_h$, which actually follows almost verbatim to that of its continuous counterpart $\mathcal{A}$ (see Lemma 3.3), and the proof can be omitted.

Lemma 4.2. *Assume that* $\kappa_1\in\big(0,2\delta_3\big(2\nu_1-\frac{\kappa_3\nu_2}{\delta_1}\big)\big)$, $\kappa_2\in(0,2\delta_2)$, $\kappa_3\in\big(0,\frac{2\nu_1\delta_1}{\nu_2}\big)$ *with* $\delta_1\in\big(0,\frac{1}{\nu_2}\big)$, $\delta_2\in\big(0,\frac{2}{\gamma}\big)$, $\delta_3\in(0,1)$. *Then, there exists* $r_0>0$ *(cf.* (3.22)*) such that for each* $r\in(0,r_0)$*, problem* (4.9) *has a unique solution* $\mathcal{A}_h(\boldsymbol{w}_h,\boldsymbol{\phi}_h):=\vec{\xi}_h\in\mathbb{H}_h$*, for each* $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\in\mathbf{H}_h$ *with* $\|\boldsymbol{w}_h\|_{1,\Omega}\le r$*. Moreover, there exists* $C_{\mathcal{A}}$*, independent of* $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)$*, such that*

$$\|\mathcal{A}_h(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\|_{\mathbb{H}_h} \;=\; \|\vec{\xi}_h\|_{\mathbb{H}_h} \;\le\; C_{\mathcal{A}}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\phi}_h\|_{0,\Omega} \qquad \forall\,(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\in\mathbf{H}_h\,.$$

We now provide the discrete version of Lemma 3.4.

Lemma 4.3. *For each* $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\in\mathbf{H}_h$*, problem* (4.10) *has a unique solution* $\boldsymbol{\varphi}_h\in\mathbf{H}^{\boldsymbol{\varphi}}_h$*, with* $\boldsymbol{\varphi}_h|_{\Gamma_{\mathrm{D}}}=\boldsymbol{\varphi}_{\mathrm{D},h}$*. Moreover, there exists a constant* $\widetilde{C}_{\mathcal{B}}>0$ *independent of* $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)$*, such that*

$$\|\mathcal{B}_h(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\|_{1,\Omega} \;=\; \|\boldsymbol{\varphi}_h\|_{1,\Omega} \;\le\; \widetilde{C}_{\mathcal{B}}\big\{\|\boldsymbol{w}_h\|_{1,\Omega}\|\boldsymbol{\phi}_h\|_{1,\Omega} \,+\, |\boldsymbol{\alpha}|\,|\boldsymbol{g}| \,+\, \|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\big\}\,. \tag{4.12}$$

Proof. Let $\boldsymbol{\varphi}_{1,h}:=\mathcal{I}^{SZ}_h(\boldsymbol{\varphi}_1)\in\mathbf{H}^{\boldsymbol{\varphi}}_h$ which satisfies $\boldsymbol{\varphi}_{1,h}|_{\Gamma_{\mathrm{D}}}=\boldsymbol{\varphi}_{\mathrm{D},h}$. Then, similar to Lemma 3.4, we consider the auxiliary discrete problem: Find $\boldsymbol{\varphi}_{0,h}\in\mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_{\mathrm{D}}}$ such that

$$\mathbf{C}(\boldsymbol{\varphi}_{0,h},\boldsymbol{\psi}_h) \;=\; \widetilde{G}_{w_h,\phi_h}(\boldsymbol{\psi}_h) \qquad \forall\,\boldsymbol{\psi}_h\in\mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_{\mathrm{D}}}\,, \tag{4.13}$$

where

$$\widetilde{G}_{w_h,\phi_h}(\boldsymbol{\psi}_h) \;:=\; -\int_\Omega(\nabla\boldsymbol{\phi}_h)\boldsymbol{w}_h\cdot\boldsymbol{\psi}_h \;-\; \int_\Omega\mathbb{K}\nabla\boldsymbol{\varphi}_{1,h}:\nabla\boldsymbol{\psi}_h \qquad \forall\,\boldsymbol{\psi}_h\in\mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_{\mathrm{D}}}\,.$$

Next, the boundedness and the ellipticity of $\mathbf{C}$ are obtained exactly as in the proof of Lemma 3.4 with the same ellipticity constant $\widetilde{\alpha}$ given by (3.31). On the other hand, reasoning as in the proof of Lemma 3.4, and applying the stability property of $\mathcal{I}^{SZ}_h$ (cf. (4.6)) and the equinormic property of $\boldsymbol{\varphi}_1$ (cf. (3.27)), we easily deduce that

$$|\widetilde{G}_{w_h,\phi_h}(\boldsymbol{\psi}_h)| \;\le\; \Big\{c_1(\Omega)\|\boldsymbol{w}_h\|_{1,\Omega}\|\boldsymbol{\phi}_h\|_{1,\Omega} \,+\, c\,\|\mathbb{K}\|_{\infty,\Omega}\,\|\boldsymbol{\varphi}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big\}\|\boldsymbol{\psi}_h\|_{1,\Omega},$$

$\forall \psi \in \mathbf{H}^{\varphi}_{h,\Gamma_D}$, which says that $\widetilde{G}_{w_h,\phi_h} \in [\mathbf{H}^{\varphi}_{h,\Gamma_D}]'$ and

$$\|\widetilde{G}_{w_h,\phi_h}\| \le c_1(\Omega)\|w_h\|_{1,\Omega}\|\phi_h\|_{1,\Omega} + c\,\|\mathbb{K}\|_{\infty,\Omega}\,\|\varphi_D\|_{1/2,\Gamma_D}.$$

Therefore, a direct application of the Lax-Milgram lemma implies that there exists a unique $\varphi_{0,h} \in \mathbf{H}^{\varphi}_{h,\Gamma_D}$ that satisfies (4.13) with

$$\|\varphi_{0,h}\|_{1,\Omega} \le \frac{1}{\widetilde{\alpha}}\left\{c_1(\Omega)\|w_h\|_{1,\Omega}\|\phi_h\|_{1,\Omega} + c\,\|\mathbb{K}\|_{\infty,\Omega}\,\|\varphi_D\|_{1/2,\Gamma_D}\right\}.$$

Then, $\varphi_h := \varphi_{0,h} + \varphi_{1,h} \in \mathbf{H}^{\varphi}_h$, which in fact satisfies that $\varphi_h|_{\Gamma_D} = \varphi_{D,h}$, is the unique solution of (4.10). In addition, the estimate (4.12) holds with $\widetilde{C}_{\mathcal{B}} := \max\{\widetilde{\alpha}^{-1}\,c_1(\Omega), \widetilde{c}\,(\widetilde{\alpha}^{-1}\,\|\mathbb{K}\|_{\infty,\Omega} + 1)\}$. □

4.2. Solvability analysis of the fixed point equation

In this section we establish the solvability of the fixed point problem (4.11) by applying the Brouwer fixed-point theorem [15, Theorem 9.9-2]. To this end, we begin with the discrete version of Lemma 3.5.

Lemma 4.4. *Given $r \in (0, r_0)$, with r_0 given by (3.22), we let*

$$W^h_r := \left\{(w_h, \phi_h) \in \mathbf{H}_h : \quad \|(w_h, \phi_h)\|_{\mathbf{H}} \le r\right\},$$

and assume that the data satisfy

$$c(r)\,|\alpha|\,|g| + \widetilde{C}_{\mathcal{B}}\,\|\varphi_D\|_{1/2,\Gamma_D} \le r \tag{4.14}$$

where $c(r) := rC_{\mathcal{A}}(1 + r\widetilde{C}_{\mathcal{B}})$, with $C_{\mathcal{A}}$ and $\widetilde{C}_{\mathcal{B}}$ as in (3.16) and (4.12), respectively. Then, there holds $\mathscr{L}_h(W^h_r) \subseteq W^h_r$.

In order to provide the discrete analogue of Lemma 3.6, we notice in advance that, instead of the regularity assumptions employed in the continuous case, which are not applicable in the present case, we simple utilize an L^4-L^4-L^2 argument.

Lemma 4.5. *Let $r \in (0, r_0)$ with r_0 given by (3.22). Then, there exists a constant $\widetilde{L}_{\mathcal{A}} > 0$, independent of r, such that for all $(w_h, \phi_h), (\widetilde{w}_h, \widetilde{\phi}_h) \in W^h_r$, with $\|w_h\|_{1,\Omega}, \|\widetilde{w}_h\|_{1,\Omega} \le r$, there holds*

$$\begin{aligned}\|\mathcal{A}_h(w_h, \phi_h) - \mathcal{A}_h(\widetilde{w}_h, \widetilde{\phi}_h)\|_{\mathbb{H}} \le \widetilde{L}_{\mathcal{A}}\Big\{&\|\mathcal{A}_{3,h}(w_h, \phi_h)\|_{1,\Omega}\,\|w_h - \widetilde{w}_h\|_{1,\Omega} \\ &+ \|\mathcal{A}_{1,h}(w_h, \phi_h)\|_{0,4,\Omega}\,\|\phi - \widetilde{\phi}\|_{0,4,\Omega} + |\alpha|\,|g|\,\|\phi_h - \widetilde{\phi}_h\|_{1,\Omega}\Big\}.\end{aligned} \tag{4.15}$$

Proof. It proceeds exactly as in the proof of Lemma 3.6, except for the derivation of the discrete analogue of (3.40), where, instead of choosing the values of p, q determined by the regularity parameter ε, it suffices to take $p = q = 2$, thus obtaining

$$|(\mathbf{A}_{\widetilde{\phi}_h} - \mathbf{A}_{\phi_h})(\vec{\xi}_h, \vec{\xi}_h - \vec{\zeta}_h)| \le 2\,L_\nu(2 + \kappa_3^2)^{1/2}\,\|t_h\|_{0,4,\Omega}\,\|\phi_h - \widetilde{\phi}_h\|_{0,4,\Omega}\,\|\vec{\xi}_h - \vec{\zeta}_h\|_{\mathbb{H}},$$

for all $(w_h, \phi_h), (\widetilde{w}_h, \widetilde{\phi}_h)$, with $\vec{\xi}_h = (t_h, \sigma_h, u_h) := \mathcal{A}_h(w_h, \phi_h) \in \mathbb{H}_h$ and $\vec{\zeta}_h = (\widetilde{t}_h, \widetilde{\sigma}_h, \widetilde{u}_h) := \mathcal{A}_h(\widetilde{w}_h, \widetilde{\phi}_h) \in \mathbb{H}_h$. Thus, since the elements of $\mathbb{H}^t_h$ are piecewise polynomials, we can guarantee that $\|t_h\|_{0,4,\Omega} < \infty$ for each $t_h \in \mathbb{H}^t_h$. The proof concludes with $\widetilde{L}_{\mathcal{A}} := \frac{2}{\alpha(\Omega)}\max\{c_1(\Omega)(2 + \kappa_3^2)^{1/2}, 2\,L_\nu(2 + \kappa_3^2)^{1/2}, \sqrt{2}\,(2 + \kappa_2^2)^{1/2}\}$. □

The discrete version of Lemma 3.7 is given as follows.

Lemma 4.6. *Let $L_{\mathcal{B}} > 0$ as in Lemma 3.7. Then, for all $(w_h, \phi_h), (\widetilde{w}_h, \widetilde{\phi}_h) \in W^h_r$ there holds*

$$\|\mathcal{B}_h(w_h, \phi_h) - \mathcal{B}_h(\widetilde{w}_h, \widetilde{\phi}_h)\|_{1,\Omega} \le L_{\mathcal{B}}\Big\{\|w_h - \widetilde{w}_h\|_{1,\Omega} + \|\phi_h - \widetilde{\phi}_h\|_{1,\Omega}\Big\}. \tag{4.16}$$

Proof. It corresponds to an adaptation of the proof of Lemma 3.7 to the discrete context. □

Now, combining Lemmas 4.5 and 4.6, and employing the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, we can prove the discrete version of Lemma 3.8.

Lemma 4.7. *Let r and W^h_r as in Lemma 4.4. Then, there exits a positive constant C, depending only on $\widetilde{L}_{\mathcal{A}}$ and $L_{\mathcal{B}}$ (cf. (4.15) and (4.16), respectively), such that for all $(w_h, \phi_h), (\widetilde{w}_h, \widetilde{\phi}_h) \in W^h_r$ there holds*

$$\begin{aligned}\|\mathscr{L}_h(w_h, \phi_h) - \mathscr{L}_h(\widetilde{w}_h, \widetilde{\phi}_h)\|_{\mathbf{H}} \le C\Big\{&\|\mathcal{A}_{3,h}(w_h, \phi_h)\|_{1,\Omega} + c_1(\Omega)\,\|\mathcal{A}_{1,h}(w_h, \phi_h)\|_{0,4,\Omega} \\ &+ |\alpha|\,|g| + L_{\mathcal{B}}\Big\}\,\|(w_h, \phi_h) - (\widetilde{w}_h, \widetilde{\phi}_h)\|_{\mathbf{H}}.\end{aligned}$$

More precisely, there holds $C = \max\{(1 + L_{\mathcal{B}})\widetilde{L}_{\mathcal{A}}, 1\}$.

Consequently, since the foregoing lemma confirms the continuity of $\mathscr{L}_h$, by a straightforward application of Brouwer fixed point theorem (cf. [15, Theorem 9.9-2]) on the convex and compact set $W^h_r \subseteq \mathbf{H}_h$, we can provide the main result of this section.

Theorem 4.8. *Suppose that the parameters κ_1, κ_2 and κ_3 satisfy the conditions required by Lemma 4.2. Let r and W_r^h as in Lemma 4.4, and assume that the data satisfy (4.14). Then, problem (4.8) has at least one solution $(\vec{\xi}_h, \boldsymbol{\varphi}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{\varphi}}$ such that $\boldsymbol{\varphi}_h|_{\Gamma_D} = \boldsymbol{\varphi}_{D,h}$, with $(\boldsymbol{u}_h, \boldsymbol{\varphi}_h) \in W_r^h$, and there holds*

$$\|\vec{\xi}_h\|_{\mathbb{H}} \le C_{\mathcal{A}}\, r\, |\boldsymbol{\alpha}|\, |\boldsymbol{g}|,$$

and

$$\|\boldsymbol{\varphi}_h\|_{1,\Omega} \le \widetilde{C}_{\mathcal{B}} \left\{ r\, \|\boldsymbol{u}_h\|_{1,\Omega} + \|\boldsymbol{\varphi}_D\|_{1/2,\Gamma_D} \right\}.$$

5. A priori error analysis

We now aim to derive the a priori error estimates for the Galerkin scheme given by (4.8). To this end, given $((\boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{u}), \boldsymbol{\varphi}) := (\vec{\xi}, \boldsymbol{\varphi}) \in \mathbb{H} \times \mathbf{H}^1(\Omega)$, with $(\boldsymbol{u}, \boldsymbol{\varphi}) \in W_r$ and $((\boldsymbol{t}_h, \boldsymbol{\sigma}_h, \boldsymbol{u}_h), \boldsymbol{\varphi}_h) := (\vec{\xi}_h, \boldsymbol{\varphi}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{\varphi}}$, with $(\boldsymbol{u}_h, \boldsymbol{\varphi}_h) \in W_r^h$ solutions of (3.6) and (4.8), respectively, we first observe that the above problems can be rewritten as two pairs of corresponding continuous and discrete formulations, namely

$$\mathbf{A}_{\boldsymbol{\varphi}}(\vec{\xi}, \vec{\boldsymbol{\eta}}) + \mathbf{B}_{\boldsymbol{u}}(\vec{\xi}, \vec{\boldsymbol{\eta}}) = F_{\boldsymbol{\varphi}}(\vec{\boldsymbol{\eta}}) \qquad \forall\, \vec{\boldsymbol{\eta}} \in \mathbb{H}, \tag{5.1a}$$

$$\mathbf{A}_{\boldsymbol{\varphi}_h}(\vec{\xi}_h, \vec{\boldsymbol{\eta}}_h) + \mathbf{B}_{\boldsymbol{u}_h}(\vec{\xi}_h, \vec{\boldsymbol{\eta}}_h) = F_{\boldsymbol{\varphi}_h}(\vec{\boldsymbol{\eta}}_h) \qquad \forall\, \vec{\boldsymbol{\eta}}_h \in \mathbb{H}_h, \tag{5.1b}$$

and

$$\mathbf{C}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = G_{\boldsymbol{u},\boldsymbol{\varphi}}(\boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \mathrm{H}^1_{\Gamma_D}(\Omega), \tag{5.2a}$$

$$\mathbf{C}(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) = G_{\boldsymbol{u}_h,\boldsymbol{\varphi}_h}(\boldsymbol{\psi}_h) \qquad \forall\, \boldsymbol{\psi}_h \in \mathbf{H}^{\boldsymbol{\varphi}}_{h,\Gamma_D}. \tag{5.2b}$$

Our goal is to obtain an upper bound for the error $\|(\vec{\xi}, \boldsymbol{\varphi}) - (\vec{\xi}_h, \boldsymbol{\varphi}_h)\|_{\mathbb{H}\times\mathbf{H}^1(\Omega)}$. For this purpose, we first recall from [36, Theorem 11.1] an abstract result that corresponds to the standard Strang Lemma for elliptic variational problems, which will be straightforwardly applied to the pair (5.1a)-(5.1b).

Lemma 5.1. *Let H be a Hilbert space, $F \in H'$, and $A : H \times H \to \mathrm{R}$ be a bounded and H-elliptic bilinear form. In addition, let $\{H_h\}_{h>0}$ be a sequence of finite dimensional subspaces of H, and for each $h > 0$ consider a bounded bilinear form $A_h : H_h \times H_h \to \mathrm{R}$ and a functional $F_h \in H_h'$. Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic, that is, there exists a constant $\beta > 0$, independent of h such that*

$$A_h(v_h, v_h) \ge \beta\, \|v_h\|_H^2 \qquad \forall\, v_h \in H_h, \quad \forall\, h > 0.$$

In turn, let $u \in H$ and $u_h \in H_h$ such that

$$A(u, v) = F(v) \quad \forall\, v \in H \qquad \text{and} \qquad A_h(u_h, v_h) = F_h(v_h) \quad \forall\, v_h \in H_h.$$

Then for each $h > 0$ there holds

$$\|u - u_h\|_H \le C \left\{ \sup_{\substack{w_h \in H_h \\ w_h \neq \mathbf{0}}} \frac{\left| F(w_h) - F_h(w_h) \right|}{\|w_h\|_H} + \inf_{\substack{v_h \in H_h \\ v_h \neq \mathbf{0}}} \left(\|u - v_h\|_H + \sup_{\substack{w_h \in H_h \\ w_h \neq \mathbf{0}}} \frac{\left| A(v_h, w_h) - A_h(v_h, w_h) \right|}{\|w_h\|_H} \right) \right\},$$

where $C := \beta^{-1} \max\{1, \|A\|\}$.

In what follows, as usual, we denote

$$\operatorname{dist}\big(\vec{\xi}, \mathbb{H}_h\big) := \inf_{\vec{\boldsymbol{\eta}}_h \in \mathbb{H}_h} \|\vec{\xi} - \vec{\boldsymbol{\eta}}_h\|_{\mathbb{H}} \qquad \text{and} \qquad \operatorname{dist}\big(\boldsymbol{\varphi}, \mathbf{H}_h^{\boldsymbol{\varphi}}\big) := \inf_{\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}} \|\boldsymbol{\varphi} - \boldsymbol{\psi}_h\|_{1,\Omega}.$$

We now derive a preliminary estimate for the error $\|\vec{\xi} - \vec{\xi}_h\|_{\mathbb{H}} = \|(\boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{t}_h, \boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{\mathbb{H}}$.

Lemma 5.2. *There exists a constant $C_{\mathrm{ST}} > 0$ independent of h, such that*

$$\begin{aligned} \|\vec{\xi} - \vec{\xi}_h\|_{\mathbb{H}} \le C_{\mathrm{ST}} \Big\{ & \operatorname{dist}\big(\vec{\xi}, \mathbb{H}_h\big) + |\boldsymbol{\alpha}|\, |\boldsymbol{g}|\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega} + \|\boldsymbol{t}\|_{\varepsilon,\Omega}\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega} \\ & + \|\boldsymbol{u}\|_{1,\Omega}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big\}. \end{aligned} \tag{5.3}$$

Proof. From Lemma 3.3 we have that the bilinear forms $\mathbf{A}_{\boldsymbol{\varphi}} + \mathbf{B}_{\boldsymbol{u}}$ and $\mathbf{A}_{\boldsymbol{\varphi}_h} + \mathbf{B}_{\boldsymbol{u}_h}$ are both bounded and uniformly elliptic, with ellipticity constant $\frac{\alpha(\Omega)}{2}$ (cf. (3.21)). In turn, $F_{\boldsymbol{\varphi}}$ and $F_{\boldsymbol{\varphi}_h}$ are linear bounded functionals in $\mathbb{H}$ and $\mathbb{H}_h$ respectively. Then, a straightforward application of Lemma 5.1 to the context given by (5.1a)-(5.1b), gives

$$\begin{aligned} \|\vec{\xi} - \vec{\xi}_h\|_{\mathbb{H}} \le C \Bigg\{ & \sup_{\substack{\vec{\boldsymbol{\eta}}_h \in \mathbb{H}_h \\ \vec{\boldsymbol{\eta}}_h \neq \mathbf{0}}} \frac{\left| F_{\boldsymbol{\varphi}}(\vec{\boldsymbol{\eta}}_h) - F_{\boldsymbol{\varphi}_h}(\vec{\boldsymbol{\eta}}_h) \right|}{\|\vec{\boldsymbol{\eta}}_h\|_{\mathbb{H}}} \\ & + \inf_{\substack{\vec{\zeta}_h \in \mathbb{H}_h \\ \vec{\zeta}_h \neq \mathbf{0}}} \left(\|\vec{\xi} - \vec{\zeta}_h\|_{\mathbb{H}} + \sup_{\substack{\vec{\boldsymbol{\eta}}_h \in \mathbb{H}_h \\ \vec{\boldsymbol{\eta}}_h \neq \mathbf{0}}} \frac{\left| (\mathbf{A}_{\boldsymbol{\varphi}} + \mathbf{B}_{\boldsymbol{u}})(\vec{\zeta}_h, \vec{\boldsymbol{\eta}}_h) - (\mathbf{A}_{\boldsymbol{\varphi}_h} + \mathbf{B}_{\boldsymbol{u}_h})(\vec{\zeta}_h, \vec{\boldsymbol{\eta}}_h) \right|}{\|\vec{\boldsymbol{\eta}}_h\|_{\mathbb{H}}} \right) \Bigg\}, \end{aligned} \tag{5.4}$$

where $C := \frac{2}{\alpha(\Omega)} \max\{1, \|\mathbf{A}_{\varphi} + \mathbf{B}_{u}\|\}$. It is important to recall here, from (3.19), that $\|\mathbf{A}_{\varphi} + \mathbf{B}_{u}\|$ depends only on κ_1, κ_2, κ_3, ν_2, γ, $c_1(\Omega)$ and $\|\boldsymbol{u}\|_{1,\Omega}$, where $\|\boldsymbol{u}\|_{1,\Omega} \leq r$. Furthermore, we now proceed to estimate each term appearing at the right-hand side of (5.4). In order to do that, we first apply the same arguments employed to obtain (3.38), to find that

$$|F_{\varphi}(\vec{\eta}_h) - F_{\varphi_h}(\vec{\eta}_h)| \;\leq\; \sqrt{2}\,(2+\kappa_2^2)^{1/2}\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega}\,\|\vec{\eta}_h\|_{\mathbb{H}}\,. \tag{5.5}$$

Next, in order to estimate the last supremum in (5.4), we add and subtract $\vec{\xi} := (\boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{u})$, we note that

$$\begin{aligned}
&(\mathbf{A}_{\varphi} + \mathbf{B}_{u})(\vec{\zeta}_h, \vec{\eta}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h})(\vec{\zeta}_h, \vec{\eta}_h)\\
&= (\mathbf{A}_{\varphi} + \mathbf{B}_{u})(\vec{\xi}, \vec{\eta}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h})(\vec{\xi}, \vec{\eta}_h) - (\mathbf{A}_{\varphi} + \mathbf{B}_{u})(\vec{\xi} - \vec{\zeta}_h, \vec{\eta}_h) + (\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h})(\vec{\xi} - \vec{\zeta}_h, \vec{\eta}_h)\\
&= (\mathbf{A}_{\varphi} - \mathbf{A}_{\varphi_h})(\vec{\xi}, \vec{\eta}_h) + \mathbf{B}_{u-u_h}(\vec{\xi}, \vec{\eta}_h) - (\mathbf{A}_{\varphi} + \mathbf{B}_{u})(\vec{\xi} - \vec{\zeta}_h, \vec{\eta}_h) + (\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h})(\vec{\xi} - \vec{\zeta}_h, \vec{\eta}_h)
\end{aligned}$$

where, applying the same approach used in (3.40) and (3.39), together with (3.41) and the continuous embedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^{n/\varepsilon}(\Omega)$ with constant $\widetilde{C}_{\varepsilon}$, and the boundedness of the bilinear forms $\mathbf{A}_{\varphi} + \mathbf{B}_{u}$ and $\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h}$, it follows that

$$\begin{aligned}
|(\mathbf{A}_{\varphi} + \mathbf{B}_{u})(\vec{\zeta}_h, \vec{\eta}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h})(\vec{\zeta}_h, \vec{\eta}_h)| \;\leq\; &\Big\{ 2\,L_{\nu} C_{\varepsilon} \widetilde{C}_{\varepsilon} (2+\kappa_3^2)^{1/2}\,\|\boldsymbol{t}\|_{\varepsilon,\Omega}\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega}\\
&+ c_1(\Omega)\,(2+\kappa_3^2)^{1/2}\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\\
+ \big(\|\mathbf{A}_{\varphi} + \mathbf{B}_{u}\| + \|\mathbf{A}_{\varphi_h} + \mathbf{B}_{u_h}\|\big)\,&\|\vec{\xi} - \vec{\zeta}_h\|_{\mathbb{H}} \Big\} \|\vec{\eta}_h\|_{\mathbb{H}}\,.
\end{aligned} \tag{5.6}$$

In this way, by replacing (5.5) and (5.6) back into (5.4), we obtain (5.3) with C_{ST} is a positive constant depending on $\alpha(\Omega)$, L_{ν}, C_{ε}, $\widetilde{C}_{\varepsilon}$, $c_1(\Omega)$, κ_1, κ_2, κ_3, ν_2, γ, and r. □

The following result presents a estimate for the error $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega}$.

Lemma 5.3. *Assume that r satisfy that*

$$r\,\frac{c_1(\Omega)}{\widetilde{\alpha}} \;\leq\; \frac{1}{2}\,, \tag{5.7}$$

where $\widetilde{\alpha}$ is defined in (3.31), and $c_1(\Omega) := \|\boldsymbol{i}\|^2$ is the boundedness constants of the continuous injection $\boldsymbol{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$. Then, there exists a constant $\widetilde{C}_{\mathrm{ST}} > 0$, independent of h, such that

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega} \;\leq\; \widetilde{C}_{\mathrm{ST}}\,\|\boldsymbol{\varphi} - \mathcal{I}_h^{SZ}(\boldsymbol{\varphi})\|_{1,\Omega} \,+\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\,, \tag{5.8}$$

where $\mathcal{I}_h^{SZ}$ denotes the Scott-Zhang interpolant operator introduced in Section 4.

Proof. We proceed similarly as in the proof of [6, Lemma 5.3]. Indeed, by applying the triangle inequality we have that

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega} \;\leq\; \|\boldsymbol{\varphi} - \boldsymbol{\psi}_h\|_{1,\Omega} \,+\, \|\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h\|_{1,\Omega}, \tag{5.9}$$

where $\boldsymbol{\psi}_h \;=\; \mathcal{I}_h^{SZ}(\boldsymbol{\varphi}) \in \mathbf{H}_h^{\varphi}$. Moreover, nothing that $\boldsymbol{\varphi}|_{\Gamma_{\mathrm{D}}} = E(\boldsymbol{\varphi}_{\mathrm{D}})|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D}}$, we can employ [38, eq. (2.17)] and (4.5), to obtain that $\boldsymbol{\psi}_h|_{\Gamma_{\mathrm{D}}} = \boldsymbol{\varphi}_{\mathrm{D},h}$. Now, utilizing the ellipticity of bilinear the form $\mathbf{C}(\cdot,\cdot)$ on $\mathbf{H}_{h,\Gamma_{\mathrm{D}}}^{\varphi}$ (see the proof of Lemma 4.3) with constant $\widetilde{\alpha}$, along with the fact that $\mathbf{C}(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h - \boldsymbol{\varphi}_h) = G_{u_h,\varphi_h}(\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h)$ and $\mathbf{C}(\boldsymbol{\varphi}, \boldsymbol{\psi}_h - \boldsymbol{\varphi}_h) = G_{u,\varphi}(\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h)$ (see (5.2)), we deduce that

$$\begin{aligned}
\widetilde{\alpha}\,\|\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h\|_{1,\Omega}^2 &\leq \mathbf{C}(\boldsymbol{\varphi}, \boldsymbol{\psi}_h - \boldsymbol{\varphi}_h) - \mathbf{C}(\boldsymbol{\varphi} - \boldsymbol{\psi}_h, \boldsymbol{\psi}_h - \boldsymbol{\varphi}_h) - \mathbf{C}(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h - \boldsymbol{\varphi}_h)\\
&\leq |G_{u,\varphi}(\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h) - G_{u_h,\varphi_h}(\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h)| \,+\, |\mathbf{C}(\boldsymbol{\varphi} - \boldsymbol{\psi}_h, \boldsymbol{\psi}_h - \boldsymbol{\varphi}_h)|\,.
\end{aligned} \tag{5.10}$$

Next, we apply the estimate (3.43) to bound the first term on the right-hand side of (5.10), whereas for second term we use the boundedness of $\mathbf{C}(\cdot,\cdot)$ (cf. (3.29)), and (5.7). Then, it follows that

$$\begin{aligned}
\|\boldsymbol{\psi}_h - \boldsymbol{\varphi}_h\|_{1,\Omega} &\leq \frac{\|\mathbb{K}\|_{\infty,\Omega}}{\widetilde{\alpha}}\,\|\boldsymbol{\varphi} - \boldsymbol{\psi}_h\|_{1,\Omega} \,+\, \frac{r c_1(\Omega)}{\widetilde{\alpha}} \Big\{ \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big\}\\
&\leq \frac{\|\mathbb{K}\|_{\infty,\Omega}}{\widetilde{\alpha}}\,\|\boldsymbol{\varphi} - \boldsymbol{\psi}_h\|_{1,\Omega} \,+\, \frac{1}{2} \Big\{ \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big\}.
\end{aligned} \tag{5.11}$$

Then, by replacing (5.11) back into (5.9), we conclude (5.8) with $\widetilde{C}_{\mathrm{ST}} := 2(1 + \widetilde{\alpha}^{-1}\|\mathbb{K}\|_{\infty,\Omega})$. □

We now proceed to combine Lemmas 5.2 and 5.3 to derive the Céa estimate for the total error

$$\|\vec{\xi} - \vec{\xi}_h\|_{\mathbb{H}} \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega}\,.$$

In fact, by replacing the estimate for $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1,\Omega}$ given by (5.8) into the right-hand side of (5.3), and using the fact that $\|\boldsymbol{u}\|_{1,\Omega} \leq r\,C_{\mathcal{A}}|\boldsymbol{\alpha}||\boldsymbol{g}|$ (cf. (3.48)) and $\|\boldsymbol{t}\|_{\varepsilon,\Omega} \leq \widehat{C}(r)|\boldsymbol{\alpha}||\boldsymbol{g}|\|\boldsymbol{\varphi}\|_{0,\Omega}$ (cf. (3.33)) along with $\|\boldsymbol{\varphi}\|_{1,\Omega} \leq r$, we find that

$$\begin{aligned}\|\vec{\xi}-\vec{\xi}_h\|_{\mathbb{H}} &\le C_{\mathrm{ST}}\,\mathrm{dist}\big(\vec{\xi},\mathbb{H}_h\big) + C_{\mathrm{ST}}\widetilde{C}_{\mathrm{ST}}\Big\{|\boldsymbol{\alpha}|\,|\boldsymbol{g}| + \|\boldsymbol{t}\|_{\varepsilon,\Omega}\Big\}\,\|\boldsymbol{\varphi}-\mathcal{I}_h^{SZ}(\boldsymbol{\varphi})\| \\ &\quad + C_{\mathrm{ST}}\Big\{|\boldsymbol{\alpha}|\,|\boldsymbol{g}| + \|\boldsymbol{t}\|_{\varepsilon,\Omega} + \|\boldsymbol{u}\|_{1,\Omega}\Big\}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega} \\ &\le C_{\mathrm{ST}}\,\mathrm{dist}\big(\vec{\xi},\mathbb{H}_h\big) + C_0\,\|\boldsymbol{\varphi}-\mathcal{I}_h^{SZ}(\boldsymbol{\varphi})\|_{1,\Omega} + C_1\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\,,\end{aligned}$$

where $C_0 := C_{\mathrm{ST}}\widetilde{C}_{\mathrm{ST}}(1+r\widehat{C}(r))\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}|$ and $C_1 := C_{\mathrm{ST}}(1+r\widehat{C}(r)+rC_{\mathcal{A}})$. In this way, assuming that the data $\boldsymbol{\alpha}$ and $\boldsymbol{g}$ satisfy that

$$C_1\,|\boldsymbol{\alpha}|\,|\boldsymbol{g}| \;\le\; \frac{1}{2}\,, \tag{5.12}$$

we can conclude that

$$\|\vec{\xi}-\vec{\xi}_h\|_{\mathbb{H}} \;\le\; 2\,C_{\mathrm{ST}}\,\mathrm{dist}\big(\vec{\xi},\mathbb{H}_h\big) + 2\,C_0\,\|\boldsymbol{\varphi}-\mathcal{I}_h^{SZ}(\boldsymbol{\varphi})\|_{1,\Omega}. \tag{5.13}$$

Consequently, we now can establish the following main result.

Theorem 5.4. *Assume that r and the data $\boldsymbol{\alpha}$ and $\boldsymbol{g}$ are sufficiently small so that (5.7) and (5.12) hold, respectively. Then, there exists a positive constant C^*, independent of h, such that*

$$\|\vec{\xi}-\vec{\xi}_h\|_{\mathbb{H}} + \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1,\Omega} \;\le\; C^*\Big\{\mathrm{dist}\big(\vec{\xi},\mathbb{H}_h\big) + \|\boldsymbol{\varphi}-\mathcal{I}_h^{SZ}(\boldsymbol{\varphi})\|_{1,\Omega}\Big\}. \tag{5.14}$$

Proof. It follows straightforwardly from the Céa estimates (5.13) and (5.8). □

In order to provide the result concerning to the theoretical rate of convergence of (4.8), we recall from [24], the approximation properties of the specific finite element subspaces introduced in Section 4.

($\mathbf{AP}_h^{\boldsymbol{t}}$) There exists $c>0$, independent of h, such that for each $s\in(0,k+1]$, and for each $\boldsymbol{t}\in\mathbb{H}^s(\Omega)\cap\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{t},\mathbb{H}_h^{\boldsymbol{t}}\big) \;\le\; c\,h^s\,\|\boldsymbol{t}\|_{s,\Omega}\,.$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) There exists $c>0$, independent of h, such that for each $s\in(0,k+1]$, and for each $\boldsymbol{\sigma}\in\mathbb{H}^s(\Omega)\cap\mathbb{H}_0(\mathbf{div};\Omega)$, with $\mathbf{div}\boldsymbol{\sigma}\in\mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \;\le\; c\,h^s\,\big\{\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega}\big\}\,.$$

($\mathbf{AP}_h^{\boldsymbol{u}}$) There exists $c>0$, independent of h, such that for each $s\in(0,k+1]$, and for each $\boldsymbol{u}\in\mathbf{H}^{s+1}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{u},\mathbf{H}_h^{\boldsymbol{u}}\big) \;\le\; c\,h^s\,\|\boldsymbol{u}\|_{s+1,\Omega}\,.$$

Finally, thanks to the approximation property of $\mathcal{I}_h^{SZ}$ given in Lemma 4.1, there exists $c>0$, independent of h, such that for each $s\in(0,k+1]$, and for each $\boldsymbol{\varphi}\in\mathbf{H}^{s+1}(\Omega)$, there holds

$$\|\boldsymbol{\varphi}-\mathcal{I}_h^{SZ}(\boldsymbol{\varphi})\|_{1,\Omega} \;\le\; c\,h^s\,\|\boldsymbol{\varphi}\|_{s+1,\Omega}\,.$$

Therefore, from the Céa estimate (5.14), employing the aforementioned approximation properties, we can establish the following result.

Theorem 5.5. *In addition to the hypotheses of Theorems 3.9, 4.8 and 5.4, assume that there exists $s>0$ such that $\boldsymbol{t}\in\mathbb{H}^s(\Omega),\boldsymbol{\sigma}\in\mathbb{H}^s(\Omega),\mathbf{div}\,\boldsymbol{\sigma}\in\mathbf{H}^s(\Omega),\boldsymbol{u}\in\mathbf{H}^{1+s}(\Omega),\boldsymbol{\varphi}\in\mathbf{H}^{1+s}(\Omega)$. Then, there exists a positive constant C, independent of h, such that with the finite element subspaces defined by (4.1), (4.2), (4.3), (4.4), there holds*

$$\|\vec{\xi}-\vec{\xi}_h\|_{\mathbb{H}} + \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1,\Omega} \le C\,h^{\min\{s,k+1\}}\Big\{\|\boldsymbol{t}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} + \|\boldsymbol{u}\|_{1+s,\Omega} + \|\boldsymbol{\varphi}\|_{1+s,\Omega}\Big\}.$$

6. Numerical results

In this section we present three numerical experiments illustrating the performance of our semi-augmented mixed-primal finite element scheme (4.8), and confirming the rates of convergence provided by Theorem 5.5. More precisely, we take the stabilization parameters κ_1, κ_2 and κ_3 as in (3.24), which satisfies the assumption of Lemma 4.2. In addition, the zero integral mean condition for tensors in the space (4.2) is imposed via a real Lagrange multiplier. In turn, the nonlinear algebraic systems obtained are solved by the fixed-point method with a tolerance of 10^{-6}, along with the Newton method for approximate the solution of (4.8a) in each fixed-point's iteration. We take as initial guess the solution of a similar linear problem (in particular, satisfying the boundary conditions for $\boldsymbol{u}_h$ and $\boldsymbol{\varphi}_h$). The numerical results presented below were obtained using a C++ code, where the corresponding linear systems arising from (4.8a) are solved using the BiCGSTAB method, whereas for (4.8b) we employ the Conjugate Gradient method as the main solver. Finally, in all experiments we let $\boldsymbol{g}=(0,-1)^{\mathrm{t}}$ be the gravitational force, and utilizing structure triangulations of the corresponding domain in 2D. Furthermore, for the first two examples we consider polynomial degrees $k\in\{0,1,2\}$, whereas we only use $k=0$ in the last example.

We now introduce some additional notation. The individual errors are denoted by:

$$\begin{aligned}&\mathsf{e}(\boldsymbol{t}) := \|\boldsymbol{t}-\boldsymbol{t}_h\|_{0,\Omega}\,, \qquad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega}\,, \qquad \mathsf{e}(\boldsymbol{u}) := \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\,, \\ &\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1,\Omega} \quad \text{and} \quad \mathsf{e}(p) := \|p-p_h\|_{1,\Omega}\,,\end{aligned}$$

where, according to (2.6) and (2.7), p_h can be computed as:

Table 1
History of convergence for Example 1.

k	h	N	$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\varphi})$	$\mathtt{r}(\boldsymbol{\varphi})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
0	0.0404	17575	7.98e-01	−−	2.68e+00	−−	1.37e+00	−−	4.13e-02	−−	1.09e-01	−−
	0.0314	28895	6.05e-01	1.10	2.09e+00	1.00	1.05e+00	1.05	3.20e-02	1.02	8.53e-02	0.98
	0.0257	43015	4.87e-01	1.09	1.71e+00	1.00	8.51e-01	1.05	2.61e-02	1.01	6.99e-02	0.99
	0.0218	59935	4.07e-01	1.07	1.45e+00	1.00	7.16e-01	1.04	2.20e-02	1.01	5.92e-02	1.00
	0.0189	79655	3.50e-01	1.06	1.25e+00	1.00	6.18e-01	1.03	1.91e-02	1.01	5.13e-02	1.00
	0.0129	170725	2.35e-01	1.04	8.54e-01	1.00	4.18e-01	1.02	1.30e-02	1.00	3.49e-02	1.00
	0.0094	316805	1.71e-01	1.02	6.27e-01	1.00	3.05e-01	1.01	9.52e-03	1.00	2.56e-02	1.00
	0.0071	562405	1.28e-01	1.01	4.70e-01	1.00	2.29e-01	1.01	7.14e-03	1.00	1.92e-02	1.00
	0.0057	878005	1.02e-01	1.00	3.76e-01	1.00	1.83e-01	1.01	5.71e-03	1.00	1.54e-02	1.00
1	0.0404	59645	5.37e-02	−−	1.77e-01	−−	8.81e-02	−−	1.45e-04	−−	7.38e-03	−−
	0.0314	98285	3.20e-02	2.07	1.07e-01	1.99	5.28e-02	2.04	8.48e-05	2.12	4.53e-03	1.94
	0.0257	146525	2.12e-02	2.05	7.18e-02	2.00	3.51e-02	2.03	5.61e-05	2.06	3.05e-03	1.96
	0.0218	204365	1.51e-02	2.04	5.14e-02	2.00	2.51e-02	2.02	4.00e-05	2.03	2.20e-03	1.97
	0.0189	271805	1.13e-02	2.03	3.86e-02	2.00	1.88e-02	2.02	3.00e-05	2.01	1.66e-03	1.98
	0.0129	583445	5.19e-03	2.02	1.80e-02	2.00	8.70e-03	2.01	1.39e-05	2.01	7.75e-04	1.98
2	0.0404	126215	2.70e-03	−−	8.82e-03	−−	3.76e-03	−−	3.25e-07	−−	3.45e-04	−−
	0.0314	208175	1.26e-03	3.04	4.15e-03	2.99	1.76e-03	3.03	1.47e-07	3.17	1.63e-04	2.99
	0.0257	310535	6.87e-04	3.03	2.28e-03	3.00	9.58e-04	3.02	7.96e-08	3.05	8.92e-05	2.99
	0.0218	433295	4.15e-04	3.01	1.36e-03	3.08	5.79e-04	3.02	4.79e-08	3.04	5.40e-05	3.01
	0.0189	576455	2.70e-04	3.00	8.83e-04	3.03	3.76e-04	3.01	3.10e-08	3.03	3.51e-05	3.00

$$p_h \;=\; -\frac{1}{n}\Big[\operatorname{tr}(\boldsymbol{\sigma}_h + (\boldsymbol{u}_h \otimes \boldsymbol{u}_h))\Big] \;+\; \frac{1}{n|\Omega|}\|\boldsymbol{u}_h\|^2_{0,\Omega}\,.$$

On the other hand as is usual, we let $\mathtt{r}(\cdot)$ be the experimental rate of convergence given by

$$\mathtt{r}(\cdot) \;:=\; \frac{\log(\mathtt{e}(\cdot)\,/\,\mathtt{e}'(\cdot))}{\log(h\,/\,h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote errors computed on two consecutive meshes of sizes h and h', respectively. In addition, N stands for the total number of degrees of freedom (unknowns) of (4.8), that is,

$$\begin{aligned} N \;:=&\; 4\times\{\text{number of nodes in } \mathcal{T}_h\} \;+\; \big\{2(k+1)+4k\big\}\times\{\text{number of edges in } \mathcal{T}_h\} \\ &+\; \big\{(k+1)(k+2)+2k(k+1)+2k(k-1)\big\}\times\{\text{number of elements in } \mathcal{T}_h\} \;+\; 1\,. \end{aligned}$$

Example 1. We first consider the square $\Omega=(0,1)^2$, and set $\Gamma_{\mathrm{D}}=\big\{(s,0),(s,1)\in\mathrm{R}^2 \;:\; 0\le s\le 1\big\}$, $\Gamma_{\mathrm{N}}=\Gamma\setminus\Gamma_{\mathrm{D}}$, $\gamma=0.1$, $\nu(\boldsymbol{x})=(x_1^2+x_2^2+1)^{-1}$ (here $\nu_1=1$ and $\nu_2=2$), $\boldsymbol{\alpha}=(0.5,1.5)^{\mathrm{t}}$, the thermal conductivity $\mathbb{K}=2\,\mathbb{I}$, and adequately manufacture the data so that the exact solution is given by the smooth functions

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \begin{pmatrix} -\sin^2(2\pi x_1)\sin(4\pi x_2) \\ \sin(4\pi x_1)\sin^2(2\pi x_2) \end{pmatrix}, \qquad p(\boldsymbol{x}) \;=\; \cos(x_1)\cos(x_2) \,-\, \sin^2(1)\,,$$

and

$$\boldsymbol{\varphi}(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1x_2 \\ \exp(x_1+x_2) \end{pmatrix},$$

for all $\boldsymbol{x}:=(x_1,x_2)^{\mathrm{t}}\in\Omega$. In Table 1, we summarize the convergence history of the finite element scheme (4.8) as applied to Example 1. We notice there that the rate of convergence $O(h^{k+1})$ predicted by Theorem 5.5 is attained by all the unknowns.

Example 2. Next, we adapt [11, Example 3], and consider the L-shaped domain $\Omega=(-1,1)^2\setminus[0,1]^2$, and set $\Gamma_{\mathrm{N}}=\big\{(s,0),(0,s)\in\mathrm{R}^2 \;:\; 0\le s\le 1\big\}$, $\Gamma_{\mathrm{D}}=\Gamma\setminus\bar{\Gamma}_N$, $\gamma=10^{-3}$, $\nu(\boldsymbol{x})=1+\exp(-x_1^2)$ (once again $\nu_1=1$ and $\nu_2=2$), $\boldsymbol{\alpha}=(1,0.5)^{\mathrm{t}}$, $\mathbb{K}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$, and adequately manufacture the data so that the exact solution is given by the smooth functions

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \begin{pmatrix} x_2^2 \\ -x_1^2 \end{pmatrix}, \qquad p(\boldsymbol{x}) \;=\; (x_1^2+x_2^2)^{1/3} - p_0\,, \qquad \text{and} \qquad \boldsymbol{\varphi}(\boldsymbol{x}) \;=\; \begin{pmatrix} \exp(-x_1^2-x_2^2) \\ \exp(-x_1x_2) \end{pmatrix},$$

for all $\boldsymbol{x}:=(x_1,x_2)^{\mathrm{t}}\in\Omega$, where $p_0\in\mathrm{R}$ is such that $\int_\Omega p=0$ holds ($p_0\approx 8.211056552903396\text{e-}01$). In addition, we remark here that the partial derivatives of p, and hence, in particular $\mathbf{div}(\boldsymbol{\sigma})$, are singular at the origin. Indeed, according to the power $1/3$, there holds $\boldsymbol{\sigma}\in\mathbb{H}^{5/3-\varepsilon}(\Omega)$ and $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{H}^{2/3-\varepsilon}(\Omega)$ for each $\varepsilon>0$. In fact, in Table 2 we present the corresponding convergence history of Example 2, where, as predicted in advance, we note that the orders $O(h^{\min\{k+1,5/3\}})$ and $O(h^{2/3})$ are attained by $(\boldsymbol{t}_h,\boldsymbol{u}_h)$ and $\boldsymbol{\sigma}_h$, respectively. Once again, the rate of convergence predicted by Theorem 5.5 is attained by all the unknowns, except for the variable $\boldsymbol{\varphi}_h$ that preserves $O(h^{k+1})$. The foregoing phenomenon could be a special feature of this example. Furthermore, the results in Example 2 suggest that our approach should certainly be strengthened with the further incorporation of an adaptive strategy based on a suitable a-posteriori error estimates. This issue will also be addressed in a forthcoming paper.

Example 3. Finally, we aim to illustrate the accuracy of our method by considering a case in which the exact solution is unknown in a time-dependent approach. More precisely, we add $\partial_t\boldsymbol{u}$ and $\partial_t\boldsymbol{\varphi}$ to the left-hand side of first and last equations of (2.1), respectively, which, together with the boundary conditions (2.4), we consider initial conditions

Table 2
History of convergence for Example 2.

k	h	N	e($\boldsymbol{t}$)	r($\boldsymbol{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\boldsymbol{u}$)	r($\boldsymbol{u}$)	e($\boldsymbol{\varphi}$)	r($\boldsymbol{\varphi}$)	e(p)	r(p)
0	0.0707	17285	5.60e-02	––	2.77e-01	––	7.07e-02	––	1.02e-01	––	4.85e-02	––
	0.0566	26855	4.48e-02	1.00	2.23e-01	0.97	5.66e-02	1.00	8.15e-02	1.00	3.88e-02	1.00
	0.0471	38525	3.73e-02	1.00	1.87e-01	0.97	4.72e-02	1.00	6.80e-02	1.00	3.23e-02	1.00
	0.0404	52295	3.20e-02	1.00	1.61e-01	0.96	4.04e-02	1.00	5.83e-02	1.00	2.77e-02	1.00
	0.0354	68165	2.80e-02	1.00	1.42e-01	0.96	3.54e-02	1.00	5.10e-02	1.00	2.42e-02	1.00
	0.0236	152645	1.87e-02	1.00	9.63e-02	0.95	2.36e-02	1.00	3.40e-02	1.00	1.61e-02	1.00
	0.0166	305495	1.32e-02	1.00	6.94e-02	0.94	1.66e-02	1.00	2.40e-02	1.00	1.14e-02	1.00
	0.0135	465575	1.07e-02	1.00	5.69e-02	0.93	1.35e-02	1.00	1.94e-02	1.00	9.22e-03	1.00
1	0.0707	58565	2.34e-04	––	2.67e-02	––	1.35e-04	––	1.45e-03	––	7.06e-04	––
	0.0566	91205	1.56e-04	1.82	2.30e-02	0.68	9.20e-05	1.71	9.30e-04	2.00	4.62e-04	1.90
	0.0471	131045	1.12e-04	1.81	2.03e-02	0.67	6.75e-05	1.69	6.46e-04	2.00	3.27e-04	1.89
	0.0404	178085	8.47e-05	1.81	1.83e-02	0.67	5.21e-05	1.69	4.75e-04	2.00	2.44e-04	1.89
	0.0354	232325	6.66e-05	1.80	1.68e-02	0.67	4.16e-05	1.68	3.63e-04	2.00	1.90e-04	1.88
	0.0236	521285	3.23e-05	1.79	1.28e-02	0.67	2.11e-05	1.68	1.62e-04	2.00	8.93e-05	1.86
2	0.0707	123845	3.00e-05	––	1.52e-02	––	3.29e-05	––	1.64e-05	––	8.29e-05	––
	0.0566	193055	2.07e-05	1.67	1.31e-02	0.67	2.27e-05	1.67	8.39e-06	3.01	5.70e-05	1.68
	0.0471	277565	1.52e-05	1.67	1.16e-02	0.67	1.67e-05	1.67	4.85e-06	3.01	4.20e-05	1.67
	0.0404	377375	1.18e-05	1.67	1.05e-02	0.67	1.29e-05	1.67	3.05e-06	3.00	3.25e-05	1.67
	0.0354	492485	9.44e-06	1.67	9.60e-03	0.67	1.04e-05	1.67	2.04e-06	3.00	2.60e-05	1.67

Table 3
History of convergence for Example 3 for $t = 0.5$.

k	h	N	e($\boldsymbol{t}$)	r($\boldsymbol{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\boldsymbol{u}$)	r($\boldsymbol{u}$)	e($\boldsymbol{\varphi}$)	r($\boldsymbol{\varphi}$)	e(p)	r(p)
0	0.0707	5845	4.47e+01	––	1.07e+01	––	5.79e-01	––	5.15e-01	––	8.63e-01	––
	0.0566	9055	3.57e+01	1.01	8.35e+00	1.10	4.54e-01	1.08	4.02e-01	1.12	6.86e-01	1.03
	0.0471	12965	2.95e+01	1.05	6.80e+00	1.13	3.61e-01	1.26	3.21e-01	1.24	5.66e-01	1.05
	0.0404	17575	2.53e+01	0.99	5.72e+00	1.12	3.08e-01	1.03	2.70e-01	1.11	4.85e-01	1.01
	0.0354	22885	2.21e+01	1.03	4.93e+00	1.11	2.68e-01	1.02	2.34e-01	1.08	4.19e-01	1.09

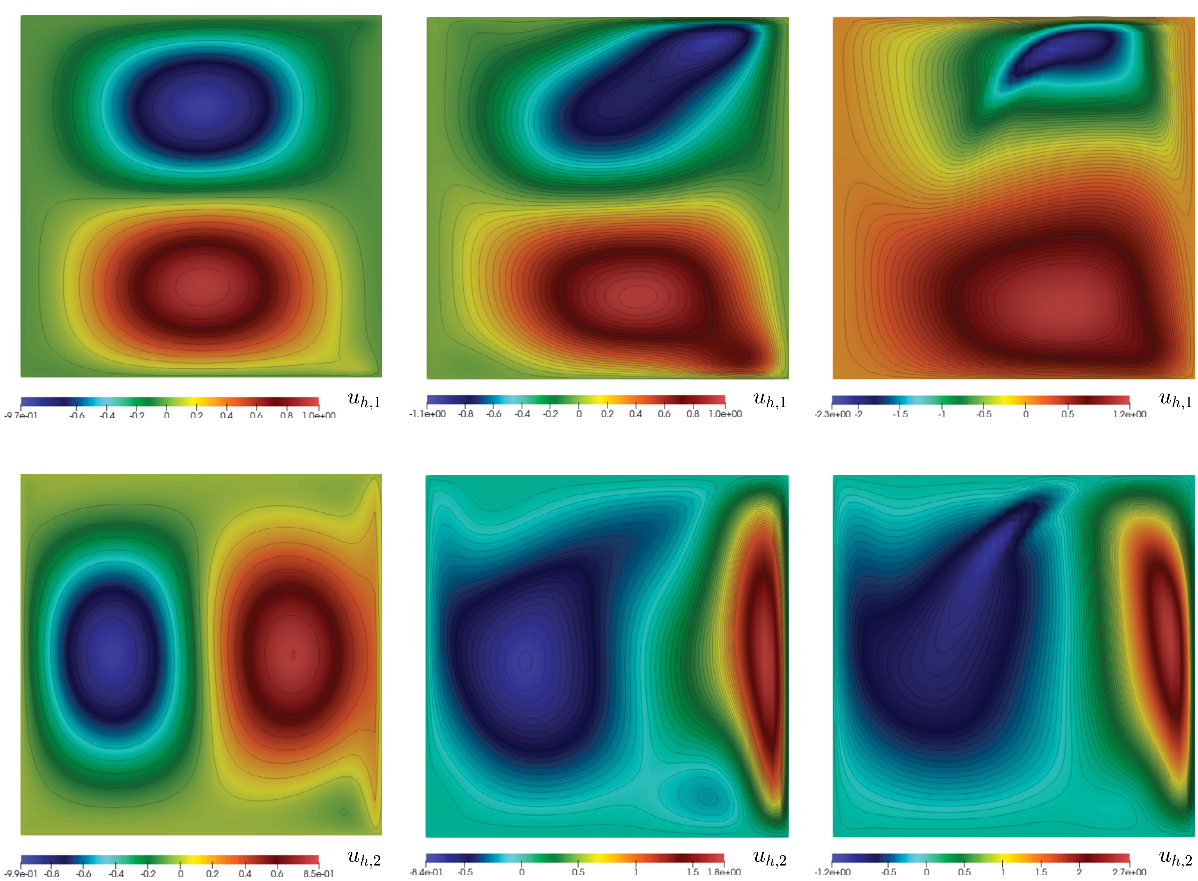

Fig. 1. Example 3, velocity components $u_{h,1}$ (top) and $u_{h,2}$ (bottom), obtained with a fully-discrete time-dependent mixed method with no manufactured analytical solution using $k = 0$ and $N = 28895$ degrees of freedom. We plot for $t \in \{t_1, t_{10}, t_{20}\}$, en each row.

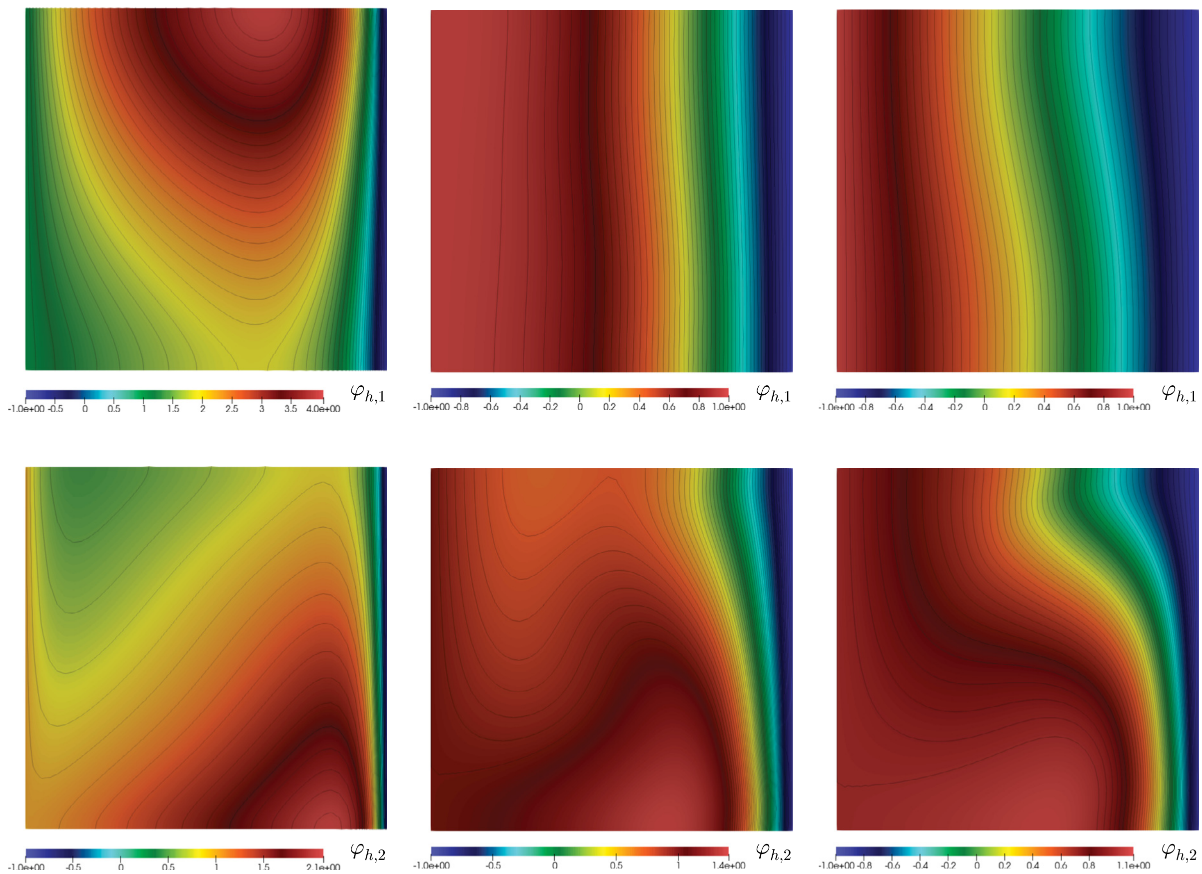

Fig. 2. Example 3, temperature $\varphi_{h,1}$ (top) and concentration $\varphi_{h,2}$ (bottom), obtained with a fully-discrete time-dependent mixed method with no manufactured analytical solution using $k=0$ and $N=28895$ degrees of freedom. We plot for $t\in\{t_1,t_{10},t_{20}\}$, en each row.

$$\boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0 \quad \text{in } \Omega \qquad \text{and} \qquad \boldsymbol{\varphi}(\cdot,0)=\boldsymbol{\varphi}_0 \quad \text{in } \Omega.$$

We remark here that, in similar way of [27], the analysis presented along of this paper can be extended to this time-dependent problem by employing backward Euler time stepping in order to obtain a fully-discrete method. On the other hand, for this example we consider once again the unit square $\Omega=(0,1)^2$, and set $\Gamma_{\mathrm{D}}=\{(0,s),(1,s)\in\mathrm{R}^2 \,:\, 0\le s\le 1\}$, $\Gamma_{\mathrm{N}}=\Gamma\setminus\Gamma_{\mathrm{D}}$, $\gamma=10^{-3}$, $\nu(\boldsymbol{x})=10^{-2}$, $\boldsymbol{\alpha}=(1,10)^{\mathrm{t}}$, $\mathbb{K}=\begin{pmatrix}1 & 0\\ 0 & 10^{-1}\end{pmatrix}$. The boundary condition is defined as

$$\boldsymbol{\varphi}_{\mathrm{D}}(\boldsymbol{x},t) = \begin{cases} (1,1)^{\mathrm{t}} & \text{if } x_1=0\\ (-1,-1)^{\mathrm{t}} & \text{if } x_1=1\end{cases}$$

for all $\boldsymbol{x}:=(x_1,x_2)^{\mathrm{t}}\in\Omega$, whereas the initial conditions are given by

$$\boldsymbol{u}_0(\boldsymbol{x}) = \begin{pmatrix}\sin^2(\pi x_1)\sin(2\pi x_2)\\ -\sin(2\pi x_1)\sin^2(\pi x_2)\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\varphi}_0(\boldsymbol{x}) = \begin{pmatrix}\exp(x_1+x_2)\\ \exp(x_1-x_2)\end{pmatrix}.$$

In addition, for the time stepping technique we use $\triangle t=\frac{1}{50}$.

In Table 3, we summarize the convergence history, where we can note that the rate of convergence $O(h^{k+1})$ predicted by Theorem 5.5 is attained by all the unknowns for $k=0$ and time step $t_{25}=0.5$. We mention that the errors and the convergence rates are computed by considering the discrete solution obtained with a finer mesh ($N=28895$) as the exact solution. Additionally, in Figs. 1 and 2, we display the approximation of the velocity components, temperature and concentration. All the figures presented there were obtained with $N=28895$ degrees of freedom (used as exact solution) in the time step $t_\ell:=\ell\cdot\triangle t$, with $\ell\in\{1,5,10,15,20\}$.

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