PROCESSES OF COMPREHENSION SHOWN BY HIGH SCHOOL TEACHERS IN SOLVING OPTIMIZATION PROBLEMS WITH THE USE OF TECHNOLOGY

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What aspects of mathematical practice are enhanced when problems that involve change or variation are approached through the use of technological tools? How does the use of particular tool (dynamic software, excel or graphing calculator) help the problem solver to represent and solve those problems? How the problem approaches that appear when using distinct tools are connected or can be complemented? These were the research questions used in this study to document the work exhibited by high school teachers when solving a set of optimization problems with the use of technology. Results indicate that the use of technology became important for teachers to represent the problem, formulate conjectures, search for relations, generalize results, and make connections.

Introduction

To what extent the use of technological tools (dynamic software, excel or graphing calculators) help high school teachers explore and solve problems that involve change or variation? What kinds of representations are favored with the use of the different technological tools? What kind of conjectures and observations do high school teachers exhibit in their problem solving approaches that involve the use of technology? In general, what type of reasoning do teachers develop in their mathematical experiences with the use of particular technological tools? These are fundamental questions that are part of the research agenda in mathematics education. Hence, the use of each tool may provide different environment and conditions for the problem solver to represent and deal with information to solve mathematical problems. As a consequence, it becomes important to document types of reasoning that problem solvers show as a result of using a particular tool. Santos (2004) states that the systematic use of dynamic software, excel, or calculators helps students represent and explore mathematical properties embedded in the problem or situation from graphic, table or algebraic representations.

> The electronic technologies – calculators and computers-are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately. They can support investigation by students in every area of mathematics, including geometry, statistics, algebra, measurement, and number. When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (NCTM, 2000, p. 24)

In this study, we are interested in analyzing high school teachers« approaches to solve series of optimization problems found in calculus textbooks. In particular, the extent to which they utilize technology to explore distinct types of representations that appear during their problem solving approaches. The research questions that directed the study were: What aspects of mathematical practices do high school teachers show while working the problems with the use of technological tools? How does the use of particular tool (dynamic software, excel or graphing

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calculator) help the problem solver to represent and solve those problems? How problem approaches that appear when using distinct tools are connected or can be complemented?

Components of a Conceptual Frame

Problem-solving has been recognized as a central activity for students to learn mathematics. In this perspective, the use of technology in problem solving environment is seen as an opportunity for students to enhance their mathematical processes that involve looking for patterns, formulating conjectures, searching for arguments and communicating results. For example, the use of dynamic software may become a powerful tool for students to represent geometrically and graphically variation phenomena without using algebraic procedures; while the use of excel may offer students the possibility of examining relevant properties of the problem through the use of a systematic list or table. Both types of representations emerge as a result of examining properties of the situation or problem from two related ways: the construction of a dynamic representation of the problem and the exploration of particular cases via the algebraic model of the situation. An integrating principle in problem–solving approaches is that students have the opportunity to pose questions around the problem that lead them to recognize relevant information needed to comprehend and explore meaning associated with concepts (Postman & Weingartner, 1969). In this context, students conceptualize their learning as a continuous activity in which they constantly formulate questions, use distinct representations, look for patterns, present conjectures, support and communicate results (Thurston, 1994).

The NTCM (2000) points out that the use technology can help students understand mathematics, but it shouldn't be used only as a means to carry out basic operations. Rather, through the use of calculators and computers students can construct more representations of the tasks and focus on examining examples or particular cases to generate conjectures and eventually pose their own problems.

> The graphic power of technological tools affords access to visual models that are powerful but that many students are unable or unwilling to generate independently. The computational capacity of technological tools extends the range of problems accessible to students and also enables them to execute routine procedures quickly and accurately, thus allowing more time for conceptualizing and modeling. (NCTM, 2000, p. 25)

According to Williamson and Kaput (1999), an important consequence of the use of technology in mathematics instruction is that it can provide conditions for students to explore relationships inductively. Thus, students tend to perceive mathematics in an experimental way (by interacting with technology) and explore relationships that eventually they need to formalize or justify. That is, mathematical ideas emerge from the processes of examining relationships in geometric, numerical and algebraic contexts. By seeing and dealing with different representations of problems, students can visualize relationships and reflect on mathematical properties that are important to solve those problems (NTCM, 2000)

The Participants, the Design, and the Procedures

Twelve high school teachers, who were taking a graduate course in mathematics education, participated in the study. The course took place during one semester and included two weekly sessions of three hours each. An important objective of the course was to work on series of tasks with the use of dynamic software, excel and calculators. It is important to mention that most of the participants in the study had never used the Cabri Géométre dynamic software or the TI 92 calculators; while most of them knew Excel or they had already worked with it.

A total of eight activities were implemented during the development of the sessions. The main topic in each activity was to solve "typical" optimization problems that appear in most of calculus textbooks, but now solved them with the help of the Cabri Géométre dynamic software, the TI 92 calculator, or Excel. Each session followed a structure that included:

i) The instructor introduced the activity to the teachers and explained to them ways to report their approaches to the problem.
ii) The participants worked on activity either individually or in pairs, using the available technological tools.
iii) The participants handed in a written report with the corresponding computer files showing their approaches to the problems, comments and extensions of the original problem. Cabri Géométre offers an option of keeping records of the problem solving processes and this information became important to analyze the teachers' work.
iv) Some participants were interviewed at the end of the course. They were asked to work on one problem and were asked to reflect on the meaning of fundamental concepts that appeared during the solution of the task.

In general, the instructional principle around this problem-solving approach is that teachers conceptualize learning as an inquiry process in which they need to formulate questions, reveal and contrast their own ideas, and present distinct arguments to communicate their results. To describe what emerged or transpired during the development of the problem sessions, we decided to focus on presenting mathematical features that we identified as crucial during teachers' interaction with one problem. Since we are not offering a detailed analysis of the teachers' performance, we decided to first identify distinct types of representations that were present during the solution of the activity. Secondly, we comment on mathematical properties that became transparent in using dynamic, numeric, and algebraic (including the use of derivative techniques) approaches to deal with the problem. And thirdly, we recognize the need and importance for teachers to examine those representations of the problems achieved through the use of technology and those that usually appear with the use of paper and pencil. That is, rather than privileging one particular approach to the problem, we take the position that it is important for teachers to move, in terms of meaning, across all those representations. In this context, all the participants have an opportunity to contrast mathematical properties that appear in paper and pencil approaches with those relationships that emerge from representing the problems through the use of technological tools. Indeed, looking at the problem from distinct perspectives seems to be an important habit that teachers can develop by using technological artifacts.

After solving the activities with the use of technology, teachers also had the opportunity of working with the same activity with the only use of a pencil and a sheet of paper. Some questions that helped organize and structure the analysis include: What problem-solving approaches do teachers exhibit while working on the optimization problems? What types of representations do they use when they work with the different technological tools, and how do they interpret them? What type of information do they identify as relevant to solve the problem? To what extent the use of different tools favors the search for connection or extensions of the problem? What types of advantages or limitations appear in representing and analyzing the task with the use of each tool?

Presentation of Results

To present mathematical features that distinguish each approach to solve the problem, we identify distinct problem solving episodes to look at mathematical properties associated with the type of representations of the problem that appear during the process of solving the task. We take only one example to illustrate the type of reasoning that emerges during the process of posing questions and ways to respond them with the use of the tools; however, the task is representative of a family of problems that high school teachers use in their calculus courses. In order to present the results we identified problem-solving episodes that include understanding the problem, construction of a model, and making sense of results. The problem used to present the results is:

From all rectangles that have a fixed area A, find the one that has the minimum perimeter.

First Episode: Understanding the Problem

What does it mean to have rectangles with a fixed area? How can we measure the perimeter of rectangles with a fixed area? How can we represent geometrically the relationship between the rectangles with fixed area and their corresponding perimeters? How can I trace the perimeter variation of those rectangles? These types of questions were initially discussed among the participants and led them to identify relevant information to construct a dynamic representation of the problem.

Second Episode: Construction of a Model

Three distinct ways to represent dynamically the problem appeared during the teachers« interaction with the problem. Mauricio and Isaac used Cabri Géométre to represent and analyze the case in which fixed area of the rectangles was 5 cm^2 (*Figure 1).*

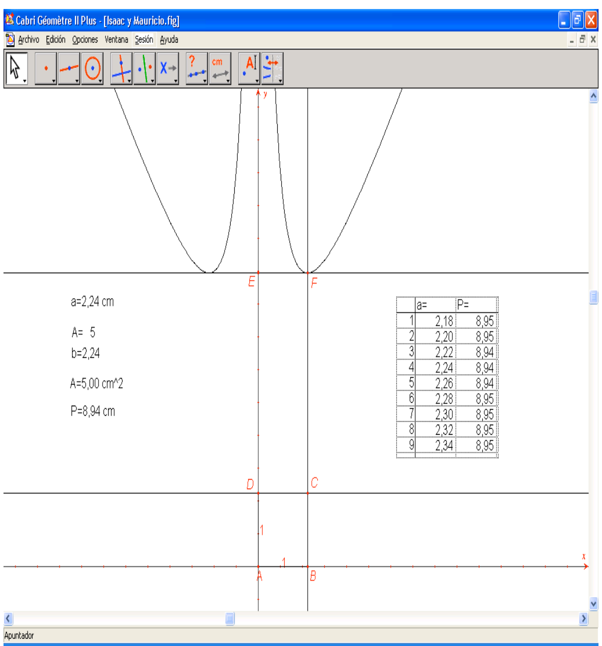


Figure 1: Dynamic approach used by Isaac and Mauricio in order to approximate the solution of a particular case.

Procedure:

1. They drew the Cartesian axes and located the origin *A*.
2. On the *x- axis* they drew segment *AB*.
3. They measured segment *AB*, which they denoted as *a (*one side of the rectangle*).*
4. They drew a perpendicular line to the *x-axis* passing by point *B*.
5. Selected 5 as the given area and calculate — to determine *b* (the other side of the rectangle).
6. They transferred value *b* on the *y- axis*, that is, *AD* = — .
7. They drew perpendicular line, to *y-axis* passing by through point *D*. This line intersected the perpendicular to *x-axis* that passes by *B* at point *C*.
8. They drew rectangle *ABCD* and calculated (using the software) its area and perimeter.
9. They transferred the perimeter's magnitude of the *ABCD* rectangle on the *y axis*, and located segment *AE* as that measure.
10. They traced a perpendicular line to the *y-axis*, through the *E* point. This straight line cut the perpendicular line to *x-axis* that passes by *B* at *F*.
11. They drew locus of point *F* point when point *B* is moved along the *x- axis*.
12. They tabulated several values of *a*, and the respective perimeter of the *ABCD* rectangle.

Third episode: Making Sense of the Results

The locus shown includes positive and negative values of *b*, what does this mean? Isaac and Mauricio argued that it was enough to pay attention to the positive values to analyze the behavior of the perimeter, since *a* and *b* represent the side of the rectangle. Any point $P(a, p)$ on the locus of *F* has the coordinates one side of the rectangle and its corresponding perimeter. When Isaac and Mauricio moved point *B* along the *x-axis*, they observed that the point on the locus that is nearest to the *x-axis* represents the point with coordinates the values of the one side of the rectangle and its perimeter. That is, the locus of point *F* helps to identify the rectangle with minimum perimeter.

When these teachers constructed a table with some values around what they identified visually as the rectangle with minimum perimeter, they realized that there were various values of the side of the rectangle that were associated with that minimum value. Here, they realized that the software was a tool to approximate the dimensions of the rectangle with perimeter minimum. In this case, when the fixed area of the rectangle was 5 squared units, they noticed that the side for the rectangle with minimum perimeter was around 2.24 units. Based on this, they made a conjecture that the minimum perimeter is reached when the rectangle becomes a square. This conjecture was later proved by using an algebraic argument.

Another pair of teachers (Marco and Pablo) relied on other type of representation, using Cabri Géométre to approach the problem (*Figure 2*).

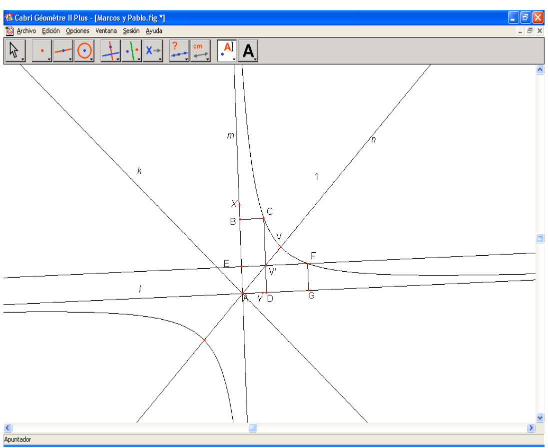


Figure 2: Dynamic construction constructed by Marcos and Pablo to deal with a particular case.

Procedure:

1. They drew line l and selected point *A* on it.
2. They drew a perpendicular line *m* to line l passing by point A. On line m they selected point *X*.
3. They drew segment *AX*.
4. They took point E on segment AX .
5. They drew segment *AE*.
6. They constructed a circumference s circumference with center in A and radius 1 unit radio. This circumference cut line l at Y point.
7. They drew line EY .
8. They drew a parallel line to line EY passing by X point. This line intersected the line l at G .
9. They constructed a perpendicular line to l passing by point *G* and a perpendicular line to *m* passing by point *E*. These two lines get intersected at point *F*.
10. They drew rectangle *AGFE*.
11. They traced the locus of point F when point *E* is moved along *AX,* that seems to be a hyperbola.
12. They drew the bisector *n* of angle EAY .
13. They identified the locus of point F as the hyperbola.
14. They noticed that line n cut the hyperbola at point V .
15. They constructed a circumference with center in A and radius AV .
16. They drew a perpendicular line *k* to line *n* passing by point *A*.
17. They located point C as the symmetric point of F with respect to line n and similarly the symmetric point *B* of point *G* with respect to line *n*.

18. They drew a perpendicular line to line *l* passing by point *C*. This line cut line *l* at point *D* .

19. They drew the rectangle *ABCD* and located V^1 as the intersection point of *CD*, *EF*, and line *n*.

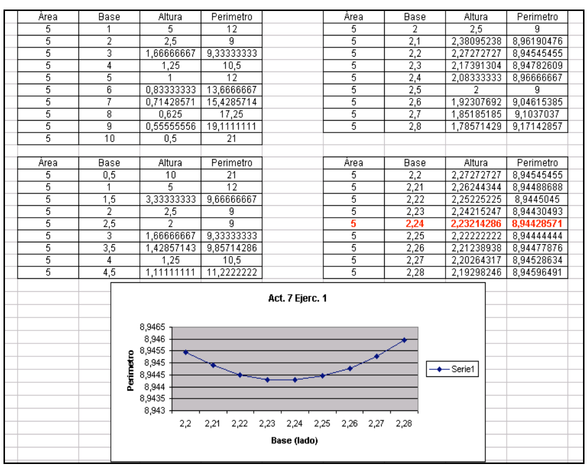
The length of segment *AX* represents the area (*A*) of the rectangles. Angle *XAG* is a right angle and *AV* is its bisector, triangles AEV^1 and ADV^1 are right triangles and isosceles. Since polygon $DAEV^1$ is a parallelogram then this polygon is also a square.

How Did They Identify the Solution? Triangles *EAY* and *XAG* are similar, that is, it is fulfilled that $\Delta EAY : \Delta XAG$, which means that = $\frac{XA}{EA} = \frac{AG}{AY}$.

Now, since *XA* represented the *A* area of the rectangles and *AY* measures 1 centimeter, then $A = AG \bullet EA$. Here, they used this information to construct all rectangles *EAGF* with fixed area *A*. When Marcos and Pablo moved point E noticed that:

> When the perimeter of rectangle ABCD increases, then the perimeter of square EADV[1] decreases, that is, there is an inverse relationship between the perimeters of those figures. Thus, when the perimeter of the square reaches its maximum value, then the perimeter of the rectangle reaches its minimum value. This happens when the vertex of the square coincides with the vertex of the hyperbola".

Yet another approach shown by Isaac and Mauricio to solve this problem involved the use of Excel *(Figure 3)*. By taking 5 squared units as the fixed area and selecting different values for the sides of the rectangles, they calculated the corresponding values of the perimeters.



Àrea	Base	Altura	Perimetro
5	1	5	12
5	2	2,5	9
5	3	1,66666667	9,33333333
5	4	1,25	10,5
5	5	1	12
5	6	0,83333333	13,6666667
5	7	0,71428571	15,4285714
5	8	0,625	17,25
5	9	0,55555556	19,1111111
5	10	0,5	21

Àrea	Base	Altura	Perimetro
5	2	2,5	9
5	2,1	2,38095238	8,96190476
5	2,2	2,27272727	8,94545455
5	2,3	2,17391304	8,94782609
5	2,4	2,08333333	8,96666667
5	2,5	2	9
5	2,6	1,92307692	9,04615385
5	2,7	1,85185185	9,1037037
5	2,8	1,78571429	9,17142857

Área	Base	Altura	Perimetro
5	0,5	10	21
5	1	5	12
5	1,5	3,33333333	9,66666667
5	2	2,5	9
5	2,5	2	9
5	3	1,66666667	9,33333333
5	3,5	1,42857143	9,85714286
5	4	1,25	10,5
5	4,5	1,11111111	11,2222222

Área	Base	Altura	Perimetro
5	2,2	2,27272727	8,94545455
5	2,21	2,26244344	8,94488688
5	2,22	2,25225225	8,9445045
5	2,23	2,24215247	8,94430493
5	**2,24**	**2,23214286**	**8,94428571**
5	2,25	2,22222222	8,94444444
5	2,26	2,21238938	8,94477876
5	2,27	2,20264317	8,94528634
5	2,28	2,19298246	8,94596491

Figure 3: Discreet approach shown by Isaac and Mauricio to approach the problem.

Figure 3 shows the process of refining values assigned to one side of the rectangles. The values included variations of one unit, 0.5 units, 0.1, and 0.01 units.

These teachers observed that while varying one side of the rectangle in 0.5 units the solution reminded on the interval (2,3) and continued refining the partition until they identified 2.24 and 2.23. Here, they represented graphically this information and visualized the solution to the problem.

During the development of the session, the participants also had opportunities to approach the problem by using algebraic approaches. Here, they confirmed results that had gotten when using the dynamic software or excel and discussed differences among those approaches.

Discussion of Results

The approaches carried out by the teachers show that the use of distinct technological tools offers the possibility of constructing various types of representations of problems that involve variation or change. Each representation can be analyzed in terms of mathematical properties that are important to comprehend concepts associated with this type of problem. For example, when using dynamic software, teachers« first goal was to think of the problem geometrically.

Segments, perpendicular lines, right triangles, symmetry and similar triangles were some basic ingredient to construct a representation that modeled the problem to observe and quantify variation dynamically. Teachers realized that they were able to construct the dynamic representation of the problems without relying on algebraic procedures. Here, they identified the potential of discussing this type of problems even with students who have not developed algebraic competences. The use of Excel became important to analyze the problem in terms of refining values of one side of the family of rectangle with a fixed area. Here, teachers recognized that the process of refining a partition is an important aspect to grasp the concept of limit. The use of algebraic procedures was seen by teachers as an opportunity to deal with the general case and as a way to verify results that emerged from using dynamic software and excel.

There is evidence that most of the teachers showed changes in using the dynamic software to represent the problem throughout the development of the sessions. Initially, they mainly focused on using commands to calculate lengths of segments, areas or perimeters of figures that appeared in the problem representation. Later, it became evident that they were also interested in comparing ratios of particular figures (lengths, perimeters, areas) and determining loci of points or figures. In addition, they became interested in providing arguments to support conjectures that emerged when moving particular elements of the problem representation. The analysis of particular cases through the use of Cabri Géométre or Excel allowed them to observe relations, patterns, results, and eventually to propose general cases.

Teachers recognized that the TI 92 calculator became a useful tool to deal with operations that involve calculating derivatives, solving equation, and graphing the function that represents the problem. Here, they were concern about the meaning associated with those operations and tried to explain connections between what they did using excel or the geometric software and their calculator work. What does it mean the derivative of function perimeter? Why does it mean to solve *f«= 0*? These were the types of questions that teachers discussed during the sessions. In this context, teachers recognized that the use of distinct tools to deal with the same problem offered them the possibility of examining fundamental concepts embedded in this type of problems from distinct angles and as a consequence to achieve a robust understanding of those concepts.

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