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Non-equilibrium Stationary Properties of the Boundary Driven Zero-Range Process with Long Jumps

Cédric Bernardin[1,2] · Patrícia Gonçalves[3] · Byron Jiménez-Oviedo[4] · Stefano Scotta[3]

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Abstract
We consider the zero-range process with long jumps and in contact with infinitely extended reservoirs in its non-equilibrium stationary state. We derive the hydrostatic limit and the Fick's law, which are a consequence of a static relationship between the exclusion process and the zero-range process. We also obtain the large deviation principle for the empirical density, i.e. we compute the non-equilibrium free energy.

Keywords Fick's law · Hydrostatics · Zero-range · Exclusion · Long-jumps · Infinitely extended reservoirs

1 Introduction

The description of the macroscopic properties of the non-equilibrium stationary state (NESS) of a large system of interacting particles driven outside of equilibrium by boundary forces has seen a lot of activity and progress recently. The results on the NESS follow from the

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✉ Cédric Bernardin
cbernard@unice.fr

Patrícia Gonçalves
pgoncalves@tecnico.ulisboa.pt

Byron Jiménez-Oviedo
byron.jimenez.oviedo@una.cr

Stefano Scotta
stefano.scotta@tecnico.ulisboa.pt

1 Université Côte d'Azur, CNRS, LJAD, Parc Valrose, 06108 Nice Cedex 02, France

2 Interdisciplinary Scientific Center Poncelet (CNRS IRL 2615), Moscow, Russia 119002

3 Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, no. 1, 1049-001 Lisbon, Portugal

4 Escuela de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Costa Rica, Heredia, Costa Rica

combination of two different approaches: one based on ad hoc exact computations of the NESS [23] and the other one, based on the so-called Macroscopic Fluctuation Theory (MFT) [15], which provides a unified beautiful treatment of non-equilibrium systems described macroscopically by diffusive or hyperbolic conservation laws. MFT is more generic, but computationally less efficient, and it is based on the development of the hydrodynamic limit theory [36, 48]. MFT is usually applied to diffusive systems, i.e. systems whose hydrodynamic equation is given by a diffusion equation, but it is, in fact, more general and the framework encapsulates also the systems whose hydrodynamic limits are described by conservation laws [1–3, 34, 41, 53]. It is also possible to use the MFT for the NESS of systems with several microscopic conservation laws, see e.g. [4, 14, 16, 52].

More recently, several studies of interacting particle systems have appeared, whose hydrodynamic limits are given by a fractional diffusion equation or a fractional conservation law [5, 10–13, 19, 27–29, 32, 45, 47]. In order to derive such equations, the authors have to invoke a suitable coarse-graining in space and time which is not the diffusive one (space does not scale as square root of time) nor the Eulerian one (space does not scale as time). Since we will only consider NESS in this paper, i.e. the system in infinite time, we prefer to not to enter in these space-time scales considerations here. In the models considered in those articles, the fractional nature is induced by the presence of non-local interactions in the microscopic dynamics (the reader can think for example of a system of independent random walks with a transition probability which has infinite variance). Since the interactions are non-local, the consequences of the boundary reservoirs is more subtle to understand, with respect to the case of local interactions, because the operators involved (microscopically and macroscopically) are non-local while a boundary condition has a local nature. One of the motivations to study such systems with non-local interactions is that they could play the role of toy models to describe some "fractional" universality classes [21, 31, 37, 38, 42, 43, 49, 50] of interacting particles with local interactions but with several conservation laws (see [6–9, 18, 33, 44] for rigorous studies).

The aim of this article is to provide a rigorous study of the NESS of a superdiffusive interacting particle system whose macroscopic behaviour is described by a fractional diffusion. Apart from [13] we are not aware of any rigorous study of the NESS for superdiffusive models. Our study focuses on the boundary driven zero-range process with long jumps. The zero-range process with finite range jumps has been introduced in [46] and has been intensively studied with different aims: existence theorems for the infinite volume dynamics, characterization of the invariant measures, derivation of hydrodynamic limits, study of phase transitions and condensation phenomena etc, [20, 22, 25, 26, 36, 39, 40, 48]. Our choice to investigate the zero-range process with long jumps instead of some other process is based on two reasons: first, it is one of the rare systems for which the NESS has some semi-explicit form; second, its NESS is strongly connected to the NESS of the boundary driven exclusion process with long jumps. The later is not explicit but it has the advantage to have "simple" hydrodynamic limits, in the sense that they are given by linear equations (but non-local and with boundary conditions), so that the hydrostatic properties of the NESS of the boundary driven exclusion process with long jumps are available. To be precise, we consider a one-dimensional superdiffusive zero-range process with long jumps and in contact with extended reservoirs at the boundary of the domain. By connecting properties of its typical profile in the NESS with the ones of the boundary driven exclusion process with long jumps, and using the fact that its NESS is of product type, we derive the form of its hydrostatic profile, we also prove a fractional Fick's law and moreover, we derive the large deviation function of the empirical density in the NESS. While the hydrodynamic limits are not derived in this work, here we provide certainly a first step in the development of the MFT for superdiffusive systems.

This paper is organised as follows. In Sect. 2 we describe the boundary driven zero range process with long jumps in contact with reservoirs (for simplicity ZRP) and the link between its NESS and the NESS of the boundary driven exclusion process with long jumps. We also present there the main results obtained in this paper. In Sect. 3 we present the proof of two results which give information about the NESS for the ZRP and are the building blocks for our main theorem. In Sect. 4 we present the proof of our main results, which are a generalization of the hydrostatic limit and the Fractional Fick's law for the ZRP. In Sect. 5 we present the proof of the Large Deviations for our model. Appendix A is dedicated to the presentation of the hydrostatic limit and Fick's law for the exclusion process with long jumps and in contact with reservoirs.

2 Models and Statement of Results

2.1 The Models

The boundary driven zero-range process with long jumps is a continuous time pure jump Markov process with countable state space $\Omega_N = \mathbb{N}_0^{\Lambda_N}$ where $\Lambda_N = \{1, \dots, N-1\}$, $N \geq 2$. A typical configuration of this process $\xi \in \Omega_N$ is denoted by $\{\xi(x)\}_{x\in\Lambda_N}$ and $\xi(x)$ represents the number of particles at site $x \in \Lambda_N$. Its dynamics is defined through a non-decreasing function $g : \mathbb{N}_0 \to [0, \infty)$ such that $g(0) = 0$ and strictly positive on the set of positive integers, and a transition probability $p : \mathbb{Z} \to [0, 1]$ given by

$$p(z) = \frac{c_\gamma}{|z|^{1+\gamma}}, \quad |z| \geq 1, \quad p(0) = 0, \tag{2.1}$$

where $0 < \gamma < 2$ and $c_\gamma = 2/\zeta(\gamma + 1) > 0$ is a normalisation factor with ζ being the Riemann zeta function. The parameter d which will appear later is defined, for $\gamma > 1$, by

$$d := d(\gamma) = \sum_{z\geq 1} z p(z) = \zeta(\gamma) \tag{2.2}$$

and corresponds to the half of the first absolute moment of $p(\cdot)$. We remark that since $0 < \gamma < 2$ the second moment of $p(\cdot)$ is infinite. Before describing the process under investigation in this article, we first describe the zero-range process with long jumps and free boundary.

2.1.1 The Zero-Range Process with Long Jumps and Free Boundary

The zero-range process $(\tilde{\xi}_t)_{t\geq 0}$ with long jumps in Λ_N, with interaction rate $g(\cdot)$, transition probability $p(\cdot)$ and free boundary conditions is the pure jump Markov process on Ω_N generated by the operator L_N^b acting on any bounded measurable function $f : \Omega_N \to \mathbb{R}$ as

$$(L_N^b f)(\xi) = \sum_{x,y\in\Lambda_N} p(y-x) g(\xi(x)) [f(\xi^{x,y}) - f(\xi)].$$

Here the configuration $\xi^{x,y}$ denotes the configuration ξ (we can always assume that $\xi(x) \geq 1$ since $g(0) = 0$) obtained by moving one particle from x to y, i.e. it is defined as

$$\xi^{x,y}(z) = (\xi(x) - 1)\mathbb{1}_{z=x} + (\xi(y) + 1)\mathbb{1}_{z=y} + \xi(z)\mathbb{1}_{z\neq x,y}. \tag{2.3}$$

The dynamics $(\tilde{\xi}_t)_{t\geq 0}$ just defined preserves the number of particles. In fact, restricted to the subspace of Ω_N composed of configurations ξ with a given fixed number of particles, the process is ergodic and it has a unique invariant measure. Therefore, it follows that on Ω_N the process has a one-parameter family of invariant measures which are defined as follows. Consider the partition function $Z : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$Z(\varphi) = \sum_{k=0}^{\infty} \frac{\varphi^k}{g(k)!},$$

where $g(k)! = g(1)\cdots g(k)$ for $k > 0$ and $g(0)! = 1$. By the ratio test it is not difficult to see that $\varphi^* := \liminf_{k\to\infty} g(k) \in (0, \infty]$ is the radius of convergence of the entire function Z. Moreover, Z is strictly increasing.

For any $\varphi < \varphi^*$ we define on Ω_N the product measure $\tilde{\nu}_\varphi$, with marginal distributions given by

$$\tilde{\nu}_\varphi\{\xi \in \Omega_N \ ; \ \xi(x) = k\} = \frac{\varphi^k}{Z(\varphi)g(k)!}. \tag{2.4}$$

A remarkable property of the zero-range process is that $\{\tilde{\nu}_\varphi \ ; \ 0 \leq \varphi < \varphi^*\}$ forms a family of invariant measures for the process generated by L_N^b. Note that for $\varphi \in [0, \varphi*)$ and any $x \in \Lambda_N$ we have that

$$E_{\tilde{\nu}_\varphi}[g(\xi(x))] = \varphi.$$

Hereinafter E_μ (resp. $\mathbb{E}_\mu$) denotes the expectation with respect to the probability measure μ (resp. the expectation with respect to the path probability measure $\mathbb{P}_\mu$ corresponding to the process starting with initial measure μ).

For every $\varphi \in [0, \varphi^*)$, we denote by $R(\varphi)$ the average number of particles per site under $\tilde{\nu}_\varphi$:

$$R(\varphi) = E_{\tilde{\nu}_\varphi}[\xi(x)].$$

The function $R(\cdot)$ can be rewritten as

$$R(\varphi) = \frac{1}{Z(\varphi)} \sum_{k=0}^{\infty} \frac{k\varphi^k}{g(k)!} = \frac{\varphi Z'(\varphi)}{Z(\varphi)} = \varphi \frac{\mathrm{d}}{\mathrm{d}\varphi}(\log(Z(\varphi))). \tag{2.5}$$

Since $R'(\varphi) = E_{\tilde{\nu}_\varphi}\left[\left(\xi(x) - E_{\nu_\varphi}[\xi(x)]\right)^2\right] > 0$, the function $R(\cdot)$ is strictly increasing from $[0, \varphi^*)$ to $[0, \infty)$, and defining

$$m^* := \lim_{\varphi \uparrow \varphi^*} R(\varphi),$$

the map $\varphi \to R(\varphi)$ is a bijection between $[0, \varphi^*)$ and $\left[0, m^*\right)$. We then denote by $\Phi(\cdot)$ the inverse map of $R(\cdot)$. Hence, we can alternatively parameterise the invariant measures by $m \in [0, m^*)$, the number the particles per site, instead of φ, i.e. we define for any $m \in [0, m^*)$ that $\nu_m = \tilde{\nu}_{\Phi(m)}$.

2.1.2 The Boundary Driven Zero-Range Process with Long Jumps

In order to define the boundary driven zero-range process with long jumps, we have now to introduce the boundary driving process. We will use infinitely extended particle reservoirs injecting or removing particles everywhere in the bulk Λ_N, so that the number of particles

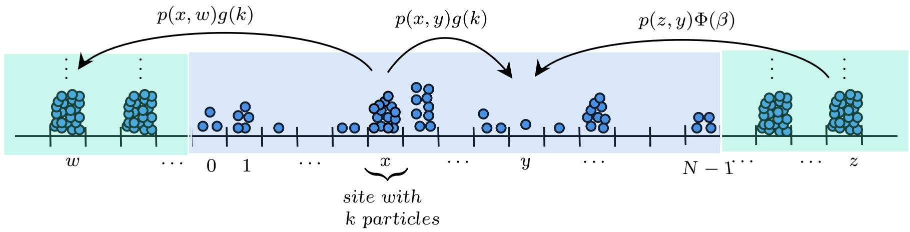

Fig. 1 Dynamics of the long-jumps zero-range process in contact with infinitely extended reservoirs

is no longer conserved (see, for example, [5, 10, 11, 13]). Any configuration $\xi \in \Omega_N$ is extended into a configuration $\xi \in (\mathbb{N}_0 \cup \{\infty\})^{\mathbb{Z}}$ by setting $\xi(z) = \infty$ if $z \notin \Lambda_N$. We also adopt the usual conventions of summation on $\mathbb{N}_0 \cup \{\infty\}$. We assume that $\alpha, \beta \in (0, m^*)$ and without loss of generality that $\alpha \leq \beta$. Observe that

$$\alpha, \beta \in (0, m^*) \iff \Phi(\alpha), \Phi(\beta) \in (0, \varphi^*). \tag{2.6}$$

The action of the generators on any bounded measurable function $f : \Omega_N \to \mathbb{R}$ at the left and right boundary are defined, respectively, by

$$\begin{aligned} (L_N^r f)(\xi) &= \sum_{x \in \Lambda_N} r_N^+\left(\tfrac{x}{N}\right) \left\{ \Phi(\beta) \left[f\left(\xi^{x,+}\right) - f(\xi) \right] + g(\xi(x)) \left[f(\xi^{x,-}) - f(\xi) \right] \right\}, \\ (L_N^\ell f)(\xi) &= \sum_{x \in \Lambda_N} r_N^-\left(\tfrac{x}{N}\right) \left\{ \Phi(\alpha) \left[f\left(\xi^{x,+}\right) - f(\xi) \right] + g(\xi(x)) \left[f(\xi^{x,-}) - f(\xi) \right] \right\}. \end{aligned} \tag{2.7}$$

Above

$$\xi^{x,\pm}(z) = (\xi(x) \pm 1)\mathbb{1}_{z=x} + \xi(z)\mathbb{1}_{z \neq x} \tag{2.8}$$

and

$$r_N^+\left(\tfrac{x}{N}\right) = \sum_{y \geq N} p(y - x) \quad \text{and} \quad r_N^-\left(\tfrac{x}{N}\right) = \sum_{y \leq 0} p(y - x). \tag{2.9}$$

The zero-range process in contact with infinitely extended reservoirs is the continuous time pure jump Markov process $(\xi_t)_{t \geq 0}$ with infinitesimal generator given by

$$L_N = \underbrace{L_N^b}_{\text{Bulk dynamics}} + \frac{\kappa}{N^\theta} \underbrace{\left(L_N^r + L_N^\ell\right)}_{\text{Boundary dynamics}} \tag{2.10}$$

where the parameters θ and κ satisfy $\theta \in \mathbb{R}$ and $\kappa > 0$. We consider the process speeded up in the time scale

$$\Theta(N) = N^\gamma \mathbb{1}_{\theta \geq 0} + N^{\gamma+\theta} \mathbb{1}_{\theta < 0} \tag{2.11}$$

so that its generator becomes $\Theta(N)L_N$ (Fig. 1).

Remark 2.1 Other models of reservoirs could have been considered but the results would be quite similar (see Sect. 2.6 in [5]).

Remark 2.2 Observe that we did not impose very restrictive conditions on the function g defining the dynamics. Since we do not have to consider the dynamics in infinite volume but only in finite volume, existence of the dynamics can be obtained by rather standard methods.

In the case of free boundaries, this is trivial because the dynamics is conservative in the number of particles so that if the initial condition has a finite number M of particles then the dynamics will evolve on the finite state space composed of configurations with M particles and thus it will be well defined. In the boundary driven case, this is less trivial because the number of particles is no longer conserved but we observe that there exists a constant C (depending on α, β and N) such that

$$L_N \left(\sum_{x \in \Lambda_N} \xi(x) \right) \leq C.$$

This implies that if we start from a configuration with M particles, then for any time horizon $T > 0$ the dynamics will be well defined and it will evolve during the time interval $[0, T]$ on the finite state space composed of particles with at most $M(T) = M + CT$ particles.

The boundary driven exclusion process with long jumps has been introduced and studied in a series of recent works [5, 10, 11, 13]. It is not the model we are interested in this paper, nevertheless, it has some links with the boundary driven zero-range process that will be crucial to establish some of our results.

2.1.3 The Boundary Driven Exclusion Process with Long Jumps

The boundary driven M-exclusion process with long jumps is the continuous time pure jump Markov process, that we denote by $(\eta_t)_{t\geq 0}$, with state space $\chi_N = \{0,1\}^{\Lambda_N}$ whose dynamics is defined as follows. A typical configuration $\eta \in \chi_N$ is denoted by $\{\eta(x)\}_{x\in\Lambda_N}$ with $\eta(x) \in \{0, 1\}$, for $x \in \Lambda_N$, where we interpret $\eta(x) = 1$ (resp. $\eta(x) = 0$) as the presence (resp. the absence) of a particle at site x. Its infinitesimal generator is given by

$$\mathcal{L}_N = \underbrace{\mathcal{L}_N^b}_{\text{Bulk dynamics}} + \frac{\kappa}{N^\theta} \underbrace{\left(\mathcal{L}_N^r + \mathcal{L}_N^\ell\right)}_{\text{Boundary dynamics}} \tag{2.12}$$

where the generator $\mathcal{L}_N^b$ corresponds to the bulk dynamics and the generators $\mathcal{L}_N^\ell$ and $\mathcal{L}_N^r$ correspond to non-conservative boundary dynamics playing the role of infinitely extended reservoirs. For $\tilde\alpha, \tilde\beta \in (0, 1)$, the action of $\mathcal{L}_N^b$, $\mathcal{L}_N^\ell$ and $\mathcal{L}_N^r$ on functions $f : \chi_N \to \mathbb{R}$ is given by

$$\begin{aligned}
(\mathcal{L}_N^b f)(\eta) &= \frac{1}{2} \sum_{x,y\in\Lambda_N} p(y-x)[f(\eta^{x,y}) - f(\eta)],\\
(\mathcal{L}_N^\ell f)(\eta) &= \sum_{x\in\Lambda_N} r_N^- \left(\tfrac{x}{N}\right) c_x(\eta; \tilde\alpha)[f(\eta^x) - f(\eta)],\\
(\mathcal{L}_N^r f)(\eta) &= \sum_{x\in\Lambda_N} r_N^+ \left(\tfrac{x}{N}\right) c_x(\eta; \tilde\beta)[f(\eta^x) - f(\eta)],
\end{aligned} \tag{2.13}$$

where $p(\cdot)$ is given by (2.1) and $r_N^\pm$ are given by (2.9). Above the configurations $\eta^{x,y}$ and η^x are defined by

$$\eta^{x,y}(z) = \eta(y)\mathbb{1}_{z=x} + \eta(x)\mathbb{1}_{z=y} + \eta(z)\mathbb{1}_{z\neq x,y} \quad \text{and} \quad \eta^x(z) = (1-\eta(x))\mathbb{1}_{z=x} + \eta(z)\mathbb{1}_{z\neq x}$$

and for any $x \in \Lambda_N$, any $\eta \in \chi_N$ and $\rho \in \{\tilde\alpha, \tilde\beta\}$ we have that

$$c_x(\eta; \rho) = [\eta(x)(1-\rho) + (1-\eta(x))\rho]. \tag{2.14}$$

When $\kappa = 0$, i.e. when the boundary reservoirs are not present, the exclusion process with long jumps conserves the number of particles $\sum_{x\in\Lambda_N}\eta(x)$ and thus, as for the zero-range, it is ergodic when restricted to the set of configurations with a fixed number of particles. As a consequence, when $\kappa = 0$, it has a one parameter family of invariant measures, which are the Bernoulli product measures with parameter $\rho \in [0, 1]$ denoted by $\{\mu_\rho\}_\rho$.

2.2 Non-equilibrium Stationary States (NESS)

2.2.1 NESS of the Boundary Driven Zero Range Process

We prove below that there exists a (unique) invariant measure, denoted by ν_{ss}^N, for the boundary driven zero-range process $(\xi_t)_{t\geq 0}$. A remarkable fact is that this NESS has a product form.

Given a function $\varphi_N : \Lambda_N \to [0, \varphi^*)$, we define on Ω_N the product probability measure $\tilde{\nu}_N := \tilde{\nu}_{\varphi_N}$ with marginal distributions given by

$$\tilde{\nu}_{\varphi_N(\cdot)}\{\xi \in \Omega_N : \xi(x) = k\} = \frac{(\varphi_N(x))^k}{Z(\varphi_N(x))g(k)!}. \tag{2.15}$$

Proposition 2.3 *For $\alpha, \beta \in (0, m^*)$ with $\alpha \leq \beta$, there exists a unique function*

$$\varphi_N := \varphi_N(\Phi(\alpha), \Phi(\beta)) : \Lambda_N \to [\Phi(\alpha), \Phi(\beta)] \subset (0, \varphi^*)$$

solving the traffic equation

$$\begin{aligned}\varphi_N(x)\Big(\sum_{y\in\Lambda_N} p(y-x) + \kappa N^{-\theta}\big(r_N^+\left(\tfrac{x}{N}\right) + r_N^-\left(\tfrac{x}{N}\right)\big)\Big)\\ = \sum_{y\in\Lambda_N}\varphi_N(y)p(y-x) + \frac{\kappa}{N^\theta}\left(\Phi(\beta)r_N^+\left(\tfrac{x}{N}\right) + \Phi(\alpha)r_N^-\left(\tfrac{x}{N}\right)\right).\end{aligned} \tag{2.16}$$

The product probability measure $\tilde{\nu}_{\varphi_N(\cdot)}$ associated to this profile φ_N coincides with the NESS ν_{ss}^N of the boundary driven zero-range process.

Remark 2.4 The assumption $\alpha, \beta \in (0, m^*)$ is crucial to establish this result. Otherwise, condensation appears and the invariant measure does not exist since mass is growing with time at the boundaries. We refer the reader to [39] for more information in the case of the boundary driven zero-range process with nearest-neighbor jumps.

2.2.2 NESS for the Exclusion Process with Long Jumps

The boundary driven exclusion process generated by $\mathcal{L}_N$ [see (2.12)] has a unique invariant measure that we will denote by μ_{ss}^N. If $\tilde{\alpha} = \tilde{\beta} = \rho$ then $\mu_{ss}^N = \mu_\rho$, the Bernoulli product measure with parameter $\rho \in [0, 1]$. Differently from the NESS ν_{ss}^N of the boundary driven zero-range process, the non-equilibrium stationary state μ_{ss}^N of the exclusion process is not product and no explicit form is known. In [5, 10, 11, 13] some macroscopic information on μ_{ss}^N has been obtained and we refer the interested reader to Appendix A.

The following proposition establishes the equality between the average of a certain observable of the zero range process in its NESS and the density of the exclusion process in its NESS.

Proposition 2.5 *Consider the boundary driven zero-range process whose generator is defined by* (2.10) *and the exclusion process whose generator is defined in* (2.13) *with*

$$\tilde{\alpha} = \frac{\Phi(\alpha)}{\Phi(\alpha) + \Phi(\beta)}, \quad \tilde{\beta} = \frac{\Phi(\beta)}{\Phi(\alpha) + \Phi(\beta)}. \tag{2.17}$$

Then, for all $x \in \Lambda_N$,

$$E_{\nu_{ss}^N}[g(\xi(x))] = (\Phi(\alpha) + \Phi(\beta))\ E_{\mu_{ss}^N}[\eta(x)] = \varphi_N(x), \tag{2.18}$$

where φ_N *is the solution of the traffic equation* (2.16).

Remark 2.6 We observe that there exists, in the case of periodic boundary conditions, also a **dynamical** mapping between the zero-range process and the exclusion process, which was first introduced in [35] for a special choice of the rate $g(k) = \mathbf{1}_{\{k>0\}}$. In that mapping, the number of particles in the zero-range process becomes the number of holes between consecutive particles in the exclusion process. This mapping holds only if the zero-range dynamics is with nearest-neighbors jumps because it is crucial to have an order of the particles in the exclusion dynamics, which is also with nearest-neighbors jumps. In [25] the author extended the previous mapping to the case where g is general (even inhomogeneous) but still with nearest-neighbors jumps. In the associated exclusion dynamics the rate to jump to the neighbor depends on the number of holes between the jumping particle and the next particle in the direction of the jump. The relationship between the two processes that we derive in this article is very different and less powerful: it is a **static** mapping in the sense of expectations as given in (2.18) and not a **dynamical** mapping, which holds only for nearest-neighbors dynamics.

2.3 Results

2.3.1 Fractional Hydrostatics

In this section we obtain the hydrostatic limit of the boundary driven ZRP, i.e. we derive the form of the macroscopic density profile of the boundary driven ZRP in its NESS. In order to do it we need to introduce the definition of (weak) solutions to the hydrostatic equations of the boundary driven exclusion process with long jumps. First we need to define some sets of test functions. To that end, for $m \in \mathbb{N}$, let $C^m([0,1])$ (resp. $C_c^m((0,1))$) be the set of all m continuously differentiable real-valued functions defined on $[0,1]$ (resp. and with compact support contained in $(0,1)$). We also use the notation $\langle \cdot, \cdot \rangle$ for the inner product in $L^2([0,1])$ and the corresponding norm is denoted by $\| \cdot \|$. Let us now introduce the operators involved in the equations and the fractional Sobolev spaces that we will deal with.

The regional fractional Laplacian $\mathbb{L}$ on the interval $[0,1]$ is the operator acting on functions $f : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \frac{|f(u)|}{(1+|u|)^{1+\gamma}} du < \infty$$

as

$$(\mathbb{L}f)(u) = c_\gamma \lim_{\epsilon \to 0} \int_0^1 \mathbb{1}_{|u-v|\geq\epsilon} \frac{f(v) - f(u)}{|u-v|^{1+\gamma}} dv, \tag{2.19}$$

for any $u \in [0,1]$ if the limit exists. We note that $\mathbb{L}f$ is well defined, if, for example, $f \in C^2([0,1])$. We also introduce the semi inner-product $\langle \cdot, \cdot \rangle_{\gamma/2}$, and the corresponding semi-norm $||\cdot||_{\gamma/2} = \langle\cdot,\cdot\rangle_{\gamma/2}$, defined by

$$\langle f, g\rangle_{\gamma/2} = \frac{c_\gamma}{2}\iint_{[0,1]^2}\frac{(f(u)-f(v))(g(u)-g(v))}{|u-v|^{1+\gamma}}dudv, \tag{2.20}$$

where $f, g : [0, 1] \to \mathbb{R}$ are functions such that $\|f\|_{\gamma/2} < \infty$ and $\|g\|_{\gamma/2} < \infty$.

Definition 2.7 Let $\mathcal{H}^{\gamma/2} := \mathcal{H}^{\gamma/2}([0, 1])$ be the Sobolev space containing all the functions $g \in L^2([0, 1])$ such that $\|g\|_{\gamma/2} < \infty$, which is a Hilbert space endowed with the norm $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ defined by

$$\|g\|^2_{\mathcal{H}^{\gamma/2}} := \|g\|^2 + \|g\|^2_{\gamma/2}.$$

If $\gamma \in (1, 2)$, by Theorem 8.2. of [24], its elements coincide a.e. with continuous functions on $[0, 1]$.

Recall (2.17). We define two functions $V_0, V_1 : (0, 1) \to (0, \infty)$ by

$$V_1(u) = r^-(u) + r^+(u) \quad \text{and} \quad V_0(u) = \tilde{\alpha}r^-(u) + \tilde{\beta}r^+(u) \tag{2.21}$$

where the functions $r^\pm : (0, 1) \to (0, \infty)$ are given by

$$r^-(u) = c_\gamma\gamma^{-1}u^{-\gamma}, \quad r^+(u) = c_\gamma\gamma^{-1}(1-u)^{-\gamma}. \tag{2.22}$$

We present now the different macroscopic equations which will appear in our study. The proof of the hydrostatic limit require to formulate these equations in a weak sense, i.e. in a distributional sense.

Definition 2.8 Let $\gamma \in (0, 2)$ and $\hat{\kappa} > 0$. We say that $\rho : [0, 1] \to [0, 1]$ is a weak solution of the stationary regional fractional reaction-diffusion equation with non-homogeneous Dirichlet boundary conditions given by

$$\begin{cases} \mathbb{L}\rho(u) + \hat{\kappa}(V_0(u) - V_1(u)\rho(u)) = 0, & \forall u \in (0, 1), \\ \rho(0) = \tilde{\alpha}, \quad \rho(1) = \tilde{\beta}, \end{cases} \tag{2.23}$$

if

(a) $\rho \in \mathcal{H}^{\gamma/2}$.
(b) $\displaystyle\int \frac{(\tilde{\alpha}-\rho(u))^2}{u^\gamma} + \frac{(\tilde{\beta}-\rho(u))^2}{(1-u)^\gamma}du < \infty$.
(c) For all $G \in C_c^\infty((0, 1))$ we have that $F_{RD}(\rho, G) := \langle\rho, \mathbb{L}G\rangle + \hat{\kappa}(\langle G, V_0\rangle) - \langle\rho, GV_1\rangle) = 0$.

Definition 2.9 Let $\gamma \in (1, 2)$. We say that $\rho : [0, 1] \to [0, 1]$ is a weak solution of the stationary regional fractional diffusion equation with non-homogeneous Dirichlet boundary condition given by

$$\begin{cases} \mathbb{L}\rho(u) = 0, & \forall u \in (0, 1), \\ \rho(0) = \tilde{\alpha}, \quad \rho(1) = \tilde{\beta}, \end{cases} \tag{2.24}$$

if

(a) $\rho \in \mathcal{H}^{\gamma/2}$.
(b) For all $G \in C_c^\infty((0, 1))$ we have that $F_{Dir}(\rho, G) := \langle\rho, \mathbb{L}G\rangle = 0$.
(c) $\rho(0) = \tilde{\alpha}$ and $\rho(1) = \tilde{\beta}$.

Remark 2.10 Since $\gamma \in (1, 2)$, as mentioned previously in Definition 2.7, if $\rho \in \mathcal{H}^{\gamma/2}$, then it coincides a.e. with a continuous function on [0, 1] so that item (c) in last definition makes sense.

Definition 2.11 Let $\gamma \in (1, 2)$ and $\hat{\kappa} > 0$. We say that $\rho : [0, 1] \to [0, 1]$ is a weak solution of the stationary regional fractional diffusion equation with fractional Robin boundary conditions

$$\begin{cases} \mathbb{L}\rho(u) = 0, \quad \forall u \in (0, 1), \\ \chi_\gamma(D^\gamma \rho)(0) = \hat{\kappa}(\tilde{\alpha} - \rho(0)), \\ \chi_\gamma(D^\gamma \rho)(1) = \hat{\kappa}(\tilde{\beta} - \rho(1)), \end{cases} \tag{2.25}$$

if

(a) $\rho \in \mathcal{H}^{\gamma/2}$.
(b) For all $G \in C_c^\infty((0, 1))$ we have that

$$F_{Rob}(\rho, G, \hat{\kappa}) := \langle \rho, \mathbb{L}G \rangle - \hat{\kappa}\left(G(0)(\tilde{\alpha} - \rho(0)) + G(1)(\tilde{\beta} - \rho(1))\right) = 0.$$

Above

$$(D^\gamma \rho)(0) = \lim_{u \to 0^+} \rho'(u)u^{2-\gamma} \quad \text{and} \quad (D^\gamma \rho)(0) = \lim_{u \to 1^-} \rho'(u)(1-u)^{2-\gamma} \tag{2.26}$$

and χ_γ is a constant defined below equation (7.4) in [30].

Definition 2.12 Let $\gamma \in (0, 2)$ and $\hat{M} \in [0, 1]$. We say that $\rho : [0, 1] \to [0, 1]$ is a weak solution of the stationary regional fractional diffusion equation with fractional Neumann boundary conditions and total mass $\hat{M}$

$$\begin{cases} \mathbb{L}\rho(u) = 0, \quad \forall u \in (0, 1), \\ (D^\gamma \rho)(0) = 0, \\ (D^\gamma \rho)(1) = 0, \end{cases} \quad , \quad \int_0^1 \rho(u)\, du = \hat{M}. \tag{2.27}$$

if

(a) $\rho \in \mathcal{H}^{\gamma/2}$.
(b) For all $G \in C_c^\infty((0, 1))$ we have that

$$F_{Neu}(\rho, G) := \langle \rho, \mathbb{L}G \rangle = 0.$$

(c) $\int_0^1 \rho(u)\, du = \hat{M}$.

Remark 2.13 Each of the previous equations has the property that their weak solutions are unique and this is a crucial feature for the proof of the hydrostatic limit. The uniqueness result is proved in detail in [5, 10, 13], apart from the Neumann case, which we present below.

Lemma 2.14 *The weak solution of* (2.27) *in the sense given above is unique and equal to* $\hat{M}$.

Proof Observe first that the constant function $\hat{M}$ is a solution. Let us consider two weak solutions ρ^1, ρ^2 of (2.27) with $\hat{\kappa} = 0$ and let $\bar{\rho} = \rho^1 - \rho^2$. Let us assume first that one can take $G = \bar{\rho}$ in $F_{Neu}(\bar{\rho}, G)$ to get $F_{Neu}(\bar{\rho}, \bar{\rho}) = \langle \bar{\rho}, \mathbb{L}\bar{\rho} \rangle = 0$. From the integration by parts formula, see, for example, Proposition 2.1 in [5], we get that $\langle \bar{\rho}, \bar{\rho} \rangle_{\gamma/2} = 0$, which implies that $\bar{\rho}$ is constant almost everywhere. From item (c) of the definition of weak solution, we conclude that $\bar{\rho}(u) = 0$ for almost everywhere $u \in [0, 1]$. Now, we just have to redo the argument by considering a sequence $\{G_k\}_{k \geq 1}$ of functions in $C_c^\infty((0, 1))$ converging to $\bar{\rho}$ with respect to the norm $\| \cdot \|_{\mathcal{H}^{\gamma/2}}$, and the proof ends. □

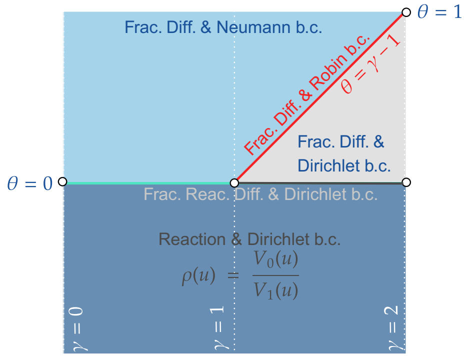

Fig. 2 Hydrostatic behavior depending on the values of θ (vertical axis) and γ (horizontal axis)

Recall that $d > 0$ is the parameter defined in (2.2) and that by Proposition 2.3 the function φ_N takes values in $[\Phi(\alpha), \Phi(\beta)]$. The following theorem is a form of hydrostatic limit for the non-equilibrium stationary boundary driven exclusion process with long jumps.

Theorem 2.15 *For any continuous function* $G : [0, 1] \to \mathbb{R}$ *and any function* $F : [\Phi(\alpha), \Phi(\beta)] \times [0, 1] \to \mathbb{R}$ *which is Lipschitz in the first component, we have that*

$$\lim_{N\to\infty} \left| \frac{1}{\#\Lambda_N} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) F\left(\varphi_N(x), \tfrac{x}{N}\right) - \int_0^1 G(u)F\left([\Phi(\alpha) + \Phi(\beta)]\,\bar\rho(u), u\right) du \right| = 0$$

where $\bar\rho : [0, 1] \to [0, 1]$ *is the measurable function defined by*

(a) *For* $\theta < 0$ *and* $\gamma \in (0, 2)$, $\bar\rho(u) = \dfrac{V_0(u)}{V_1(u)}$.
(b) *For* $\theta = 0$ *and* $\gamma \in (0, 2) - \{1\}$, $\bar\rho(\cdot)$ *is the unique weak solution of* (2.23), *with* $\hat\kappa = \kappa$.
(c) *For* $\theta \in (0, \gamma - 1)$ *and* $\gamma \in (1, 2)$, $\bar\rho(\cdot)$ *is the unique weak solution of* (2.24).
(d) *For* $\theta = \gamma - 1$ *and* $\gamma \in (1, 2)$, $\bar\rho(\cdot)$ *is the weak solution of* (2.25) *with* $\hat\kappa = \kappa d$.
(e) *For* $\theta > 0$ *and* $\gamma \in (0, 1]$ *or for* $\theta > \gamma - 1$ *and* $\gamma \in (1, 2)$, $\bar\rho(\cdot)$ *is the weak solution of* (2.25) *with* $\hat\kappa = 0$.

Moreover the profile $\bar\rho(\cdot)$ *takes values in* $[\tilde\alpha, \tilde\beta]$.

Above, # is the counting measure.

Last theorem, proved in Sect. 4, permits to prove the hydrostatics, in mean[1], of the boundary driven ZRP.

Corollary 2.16 (Hydrostatic limit in mean)

For any continuous function $G : [0, 1] \to \mathbb{R}$, *we have that*

$$\lim_{N\to\infty} \left| \frac{1}{\#\Lambda_N} \sum_{x\in\Lambda_N} G(\tfrac{x}{N}) E_{\nu_{ss}^N}[\xi(x)] - \int_0^1 G(u)\bar m(u)du \right| = 0$$

where the hydrostatic profile $\bar m(\cdot)$ *of the boundary driven zero-range process with long jumps is defined by*

$$\forall u \in [0, 1], \quad \bar m(u) = R\left[(\Phi(\alpha) + \Phi(\beta))\,\bar\rho(u)\right] \tag{2.28}$$

and $\bar\rho(\cdot)$ *is the hydrostatic profile of the boundary driven exclusion process with long jumps given in Theorem* 2.15.

[1] A sequence of integrable random variables $(X_N)_N$ is said to converge in mean if $\{\mathbb{E}[X_N)]\}_N$ converges.

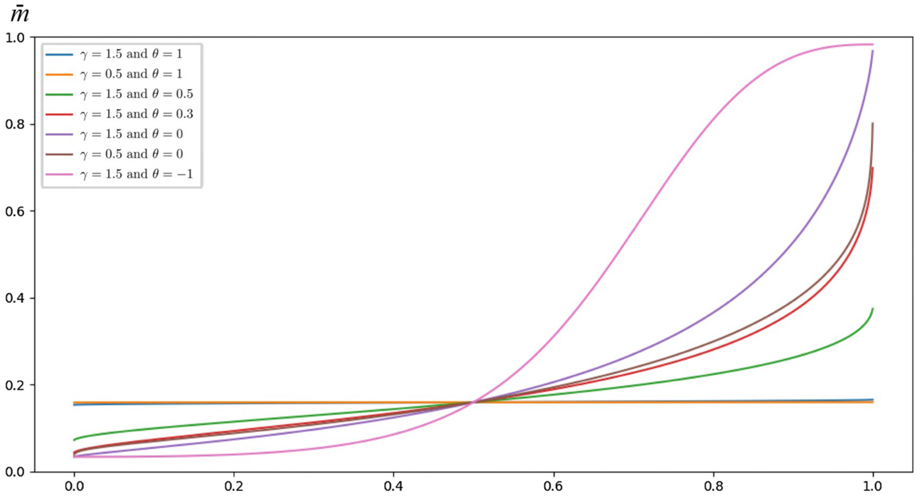

Fig. 3 Stationary profiles of the boundary driven zero-range process with long-jumps, with $\alpha = 0.2$, $\beta = 0.8$, $g(k) = \left(1 + \frac{3}{k}\right)^3$ if $k \neq 0$ and $g(0) = 0$. All the profiles $\bar{m}$ are such that $\bar{m}(1/2) = R\left(\frac{\Phi(\alpha)+\Phi(\beta)}{2}\right)$ [which follows from (3.2) evaluated for $x = N/2$]

Proof Recall first that because of the form of ν_{ss}^N we have that

$$E_{\nu_{ss}^N}[\xi(x)] = R(\varphi_N(x)) \tag{2.29}$$

for any $x \in \Lambda_N$. We apply Theorem 2.15 with the function $F : [\Phi(\alpha), \Phi(\beta)] \times [0, 1] \to \mathbb{R}$ defined by

$$F(\varphi, u) = R(\varphi).$$

This function is Lipschitz in the first variable since R is analytic on $(0, \varphi^*)$ and $[\Phi(\alpha), \Phi(\beta)] \subset (0, \varphi^*)$. Hence, for any continuous function $G : [0, 1] \to \mathbb{R}$ we have that

$$\lim_{N\to\infty} \left| \frac{1}{\#\Lambda_N} \sum_{x\in\Lambda_N} G(\tfrac{x}{N}) R(\varphi_N(x)) - \int_0^1 G(u) R\left([\Phi(\alpha) + \Phi(\beta)]\,\bar{\rho}(u)\right) du \right| = 0.$$

From (2.29) we conclude the proof. □

Remark 2.17 In Fig. 2 the profile $\bar{m}(\cdot)$ is plotted. Observe that in some range of the parameters ($\gamma < 2$, $\theta \geq 0$) the profile is non-differentiable at the boundaries. An open question is to determine its exact behaviour there.

2.3.2 Fractional Fick's Law

Our second result is the following "fractional Fick's law". For $x \in \Lambda_N \cup \{N\}$ and a configuration ξ, we denote the current over the value $x - \frac{1}{2}$ by $W_x^{zr}(\xi)$ and we define it as the rate of particles crossing $x - \frac{1}{2}$ from left to right minus the rate of particles crossing $x - \frac{1}{2}$ from right to left. Therefore, the current can be written as

$$\begin{aligned} W_x^{zr}(\xi) = & \sum_{\substack{1\le y\le x-1\\ x-1<z\le N-1}} p(z-y)(g(\xi(y)) - g(\xi(z))) \\ & + \frac{\kappa}{N^\theta}\Big[\sum_{x\le z\le N-1} r_N^-(\tfrac{z}{N})(\Phi(\alpha) - g(\xi(z))) - \sum_{1\le y\le x-1} r_N^+(\tfrac{y}{N})(\Phi(\beta) - g(\xi(y)))\Big]. \end{aligned} \tag{2.30}$$

Moreover, changing in the last definition $g(\xi(\cdot))$ by $\eta(\cdot)$, $\Phi(\alpha)$ by $\tilde{\alpha}$ and $\Phi(\beta)$ by $\tilde{\beta}$ we obtain the definition of the current for the exclusion process (see (A.14)). We will denote the current for the exclusion process by $W_x^{ex}(\eta)$. From Proposition 2.5 it is not difficult to see that

$$E_{\nu_{ss}^N}[W_x^{zr}] = (\Phi(\alpha) + \Phi(\beta))\ E_{\mu_{ss}^N}[W_x^{ex}].$$

Therefore, it is sufficient to study the behaviour of the average current for the boundary driven exclusion process with long jumps. From Theorem A.3 we can derive the next result (Fig. 3).

Theorem 2.18 (Fractional Fick's law) *Let $\bar{m}(\cdot)$ be the hydrostatic profile of the boundary driven ZRP defined in Corollary 2.16. For $u \in (0,1)$ the following fractional Fick's law holds, apart from the case $\theta = 0$ and $\gamma = 1$:*

(a) for $\theta < 0$,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N^{1-\theta-\gamma}} E_{\nu_{ss}^N}\left[W_{[uN]}^{zr}\right] &= \kappa \int_u^1 (\Phi(\alpha) - \Phi(\bar{m}(v)))r^-(v)dv \\ &\quad - \kappa \int_0^u (\Phi(\beta) - \Phi(\bar{m}(v))r^+(v)dv \\ &= \kappa c_\gamma \gamma^{-1} \int_0^1 \frac{\Phi(\alpha) - \Phi(\beta)}{v^\gamma + (1-v)^\gamma} dv; \end{aligned} \tag{2.31}$$

(b) for $\theta = 0$,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N^{1-\gamma}} E_{\nu_{ss}^N}\left[W_{[uN]}^{zr}\right] &= c_\gamma \int_0^u \int_u^1 \frac{\Phi(\bar{m}(v)) - \Phi(\bar{m}(w))}{(w-v)^{1+\gamma}} dw dv \\ &\quad + \kappa \int_u^1 (\Phi(\alpha) - \Phi(\bar{m}(v)))r^-(v)dv \\ &\quad - \kappa \int_0^u (\Phi(\beta) - \Phi(\bar{m}(v)))r^+(v)dv; \end{aligned} \tag{2.32}$$

(c) for $\theta > 0$,

$$\lim_{N\to\infty} \frac{1}{N^{1-\gamma}} E_{\nu_{ss}^N}\left[W_{[uN]}^{zr}\right] = c_\gamma \int_0^u \int_u^1 \frac{\Phi(\bar{m}(v)) - \Phi(\bar{m}(w))}{(w-v)^{1+\gamma}} dw dv. \tag{2.33}$$

Remark 2.19 In the "Neumann case", i.e. $\gamma \in (1,2)$ and $\theta > \gamma - 1$ or $\gamma \in (0,1]$ and $\theta > 0$, $\bar{m}$ is constant and the current vanishes as expected.

2.3.3 Static Large Deviations

We want to obtain the large deviation principle associated to the hydrostatic results. More precisely, we want to estimate the probability of a deviation from the typical profile which satisfies the hydrostatic equation, but remains close to some prescribed path. In order to do this, we consider a perturbation of the system. First we need to introduce some notation.

Let $\mathcal{M}$ be the space of finite signed Borel measures on $[0, 1]$. It is known that $\mathcal{M}$ is the topological dual of $C^0([0, 1])$, when the latter is equipped with the uniform convergence. Then $\mathcal{M}$ equipped with the weak-$\star$ topology [2] is a Banach space. Let $\mathcal{M}^+ \subset \mathcal{M}$ be the cone of positive measures. For any $\xi \in \Omega_N$ the empirical measure $\pi^N(\xi, du) \in \mathcal{M}^+$ is defined by

$$\pi^N(\xi, du) = \frac{1}{\#\Lambda_N} \sum_{x\in\Lambda_N} \xi_x \delta_{\frac{x}{N}}(du) \tag{2.34}$$

where δ_u is the Dirac mass on $u \in [0, 1]$. We assume that ξ is distributed according to ν_{ss}^N and to simplify we denote $\pi^N(\xi, du)$ by $\pi^N(du)$. The action of $\pi^N \in \mathcal{M}^+$ on a continuous function $G : [0, 1] \to \mathbb{R}$ is denoted by

$$\langle \pi^N, G\rangle := \int_{[0,1]} G(u)\pi^N(du) = \frac{1}{\#\Lambda_N} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) \xi_x .$$

We also define the functional $\Lambda : C^0([0, 1]) \to \mathbb{R}$ by

$$\forall G \in C^0([0, 1]), \quad \Lambda(G) = \int_0^1 \log\left(\frac{Z(e^{G(u)}\Phi(\bar{m}(u)))}{Z(\Phi(\bar{m}(u)))}\right) du$$

where $\bar{m}$ is the hydrostatic profile defined in Corollary 2.16. Its Legendre transform $\Lambda^* : \mathcal{M} \to \mathbb{R}$ is given by

$$\forall \pi \in \mathcal{M}, \quad \Lambda^*(\pi) = \sup_{G\in C^0([0,1])} \{\langle \pi, G\rangle - \Lambda(G)\} .$$

The functional Λ^* can be computed more explicitly. If π is not absolutely continuous with respect to the Lebesgue measure then it is easy to show that $\Lambda^*(\pi) = \infty$. If $\pi(du) := \pi(u)du$ is absolutely continuous with respect to the Lebesgue measure then, from (2.5) and the fact that $\Phi(\cdot)$ is the inverse function of $R(\cdot)$, we get that

$$\Lambda^*(\pi) = \int_0^1 \left\{\pi(u) \log\left(\frac{\Phi(\pi(u))}{\Phi(\bar{m}(u))}\right) - \log\left(\frac{Z(\Phi(\pi(u)))}{Z(\Phi(\bar{m}(u)))}\right)\right\} du.$$

Theorem 2.20 (Large deviations) *If $\varphi^* = +\infty$, then the sequence of random variables $\{\pi^N\}_{N\geq 1}$ satisfies a Large Deviation Principle at speed N with the good rate function Λ^*.*

3 Proof of Propositions 2.3 and 2.5

Proof of Proposition 2.3 Observe that (2.16) is a finite-dimensional linear equation which can be written in a matrix form. Let us now introduce some notation. We define the vector

[2] We recall that a sequence $\{\mu_n\}_n \in \mathcal{M}$ converges $\star$-weakly to $\mu \in \mathcal{M}$ if, and only if, for all $G \in C([0, 1])$ we have that $\int_0^1 G(x)d\mu_n(x)$ converges to $\int_0^1 G(x)d\mu(x)$. This coincides with the notion of 'weak convergence' used in probability theory.

$\varphi_N = [\varphi_N(x)]_{x\in\Lambda_N}$, the square matrix of size $N-1$ denoted by $P_N = [p(y-x)]_{(x,y)\in\Lambda_N^2}$, the diagonal matrix D_N of size $N-1$ whose diagonal elements are given by

$$\left[\sum_{y\in\Lambda_N} p(y-x) + \kappa N^{-\theta}\left(r_N^+\left(\tfrac{x}{N}\right) + r_N^-\left(\tfrac{x}{N}\right)\right)\right]_{x\in\Lambda_N},$$

and finally, the vector $R_N = \left[\Phi(\beta)r_N^+\left(\frac{x}{N}\right) + \Phi(\alpha)r_N^-\left(\frac{x}{N}\right)\right]_{x\in\Lambda_N}$. With this notation we can rewrite the traffic equation (2.16) as

$$(D_N - P_N)\varphi_N = R_N.$$

Since

$$\left|\sum_{y\in\Lambda_N} p(y-x) + \kappa N^{-\theta}\left(r_N^+\left(\tfrac{x}{N}\right) + r_N^-\left(\tfrac{x}{N}\right)\right)\right| > \sum_{y\in\Lambda_N}|p(y-x)|,$$

the matrix $(D_N - P_N)$ is strictly diagonally dominant, hence it is invertible. From this, we get that the traffic equation (2.16) has a unique solution.
We have now to show that φ_N takes values in $[\Phi(\alpha), \Phi(\beta)] \subset [0, \varphi^*)$. Let $\varphi = \max_{x\in\Lambda_N}\varphi_N(x)$ and let $x_0 \in \Lambda_N$ be such that $\varphi = \varphi_N(x_0)$. Evaluating (2.16) at $x = x_0$ and using the fact that for any $y \in \Lambda_N$, $\varphi_N(y) \le \varphi = \varphi_N(x_0)$ we get that

$$\varphi \le \frac{\Phi(\beta)r_N^+\left(\frac{x_0}{N}\right) + \Phi(\alpha)r_N^-\left(\frac{x_0}{N}\right)}{r_N^+\left(\frac{x_0}{N}\right) + r_N^-\left(\frac{x_0}{N}\right)} \le \Phi(\beta) < \varphi^* \tag{3.1}$$

where the penultimate inequality follows from the assumption $\alpha \le \beta$ and the fact that Φ is increasing, while the last inequality is due to the assumption (2.6). A similar argument shows that

$$\min_{x\in\Lambda_N}\varphi_N(x) \ge \Phi(\alpha) > 0.$$

It follows that $\tilde{\nu}_{\varphi_N(\cdot)}$ is a well defined probability measure, since $Z(\varphi_N(x))$ is finite for every $x \in \Lambda_N$.

In order to prove that $\tilde{\nu}_{\varphi_N(\cdot)}$ is the invariant measure of the boundary driven zero-range process with long jumps, we have to prove that for any bounded function $f : \Omega_N \to \mathbb{R}$, it holds

$$\int_{\Omega_N}(L_N f)(\xi)\, d\tilde{\nu}_{\varphi_N(\cdot)}(\xi) = 0.$$

Observe that from the definition of the generator we have that

$$\begin{aligned}\int_{\Omega_N}(L_N f)(\xi)d\nu_N(\xi) = \sum_{\xi\in\Omega_N}\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\Bigg\{&\sum_{x\in\Lambda_N}\sum_{y\in\Lambda_N}p(y-x)g(\xi(x))\left[f(\xi^{x,y})-f(\xi)\right]\\ &+\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}\sum_{y\geq N}p(y-x)\Phi(\beta)\left[f(\xi^{y,x})-f(\xi)\right]\\ &+g(\xi(x))\left[f(\xi^{x,y})-f(\xi)\right]\\ &+\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}\sum_{y\leq 0}p(y-x)\Phi(\alpha)\left[f(\xi^{y,x})-f(\xi)\right]\\ &+g(\xi(x))\left[f(\xi^{x,y})-f(\xi)\right]\Bigg\}\end{aligned}$$

Performing the change of variables $\xi\to\xi^{x,y}$, we get

$$\begin{aligned}&\int_{\Omega_N}(L_N f)(\xi)d\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\\ &=\sum_{\xi\in\Omega_N}f(\xi)\Bigg\{\sum_{x\in\Lambda_N}\sum_{y\in\Lambda_N}p(y-x)\left[g(\xi(x)+1)\tilde{\nu}_{\varphi_N(\cdot)}(\xi^{y,x})-g(\xi(x))\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\right]\\ &\quad+\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}\sum_{y\geq N}p(y-x)\left[g(\xi(x)+1)\tilde{\nu}_{\varphi_N(\cdot)}(\xi^{y,x})-g(\xi(x))\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\right]\\ &\quad+\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}\sum_{y\leq 0}p(y-x)\left[g(\xi(x)+1)\tilde{\nu}_{\varphi_N(\cdot)}(\xi^{y,x})-g(\xi(x))\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\right]\\ &\quad+\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}\sum_{y\geq N}p(y-x)\Phi(\beta)\left[\tilde{\nu}_{\varphi_N(\cdot)}(\xi^{x,y})-\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\right]\\ &\quad+\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}\sum_{y\leq 0}p(y-x)\Phi(\alpha)\left[\tilde{\nu}_{\varphi_N(\cdot)}(\xi^{x,y})-\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\right]\Bigg\}.\end{aligned}$$

Since $\tilde{\nu}_{\varphi_N(\cdot)}$ is a product measure we have that

$$\tilde{\nu}_{\varphi_N(\cdot)}(\xi^{y,x})=\begin{cases}\dfrac{\tilde{\nu}_{\varphi_N(\cdot)}(\xi)g(\xi(y))\varphi_N(x)}{g(\xi(x)+1)\varphi_N(y)}, & \text{if } x,y\in\Lambda_N,\\[2ex] \dfrac{\tilde{\nu}_{\varphi_N(\cdot)}(\xi)\varphi_N(x)}{g(\xi(x)+1)}, & \text{if } x\in\Lambda_N,\ \ y\notin\Lambda_N.\\[2ex] \dfrac{\tilde{\nu}_{\varphi_N(\cdot)}(\xi)g(\xi(y))}{\varphi_N(y)}, & \text{if } y\in\Lambda_N,\ \ x\notin\Lambda_N.\end{cases}$$

Therefore,

$$\int_{\Omega_N} (L_N f)(\xi) d\tilde{\nu}_{\varphi_N(\cdot)}(\xi) = \sum_{\xi\in\Omega_N} f(\xi)\tilde{\nu}_{\varphi_N(\cdot)}(\xi) \left\{ \sum_{x\in\Lambda} \frac{g(\xi(x))}{\varphi_N(x)} \right.$$
$$\times \left[\sum_{y\in\Lambda_N} p(y-x)\varphi_N(y) + \frac{\kappa}{N^\theta}\left[\Phi(\beta) r_N^+\left(\tfrac{x}{N}\right) + \Phi(\alpha) r_N^-\left(\tfrac{x}{N}\right)\right] \right.$$
$$\left.\left. -\varphi_N(x) \left(\sum_{y\in\Lambda_N} p(y-x) + \kappa N^{-\theta}\left(r_N^+\left(\tfrac{x}{N}\right) + r_N^-\left(\tfrac{x}{N}\right)\right)\right)\right]\right\}$$
$$+ \frac{\kappa}{N^\theta} \sum_{\xi\in\Omega_N} f(\xi)\tilde{\nu}_{\varphi_N(\cdot)}(\xi) \left\{ \sum_{x\in\Lambda} (\varphi_N(x) - \Phi(\alpha))\, r_N^-\left(\tfrac{x}{N}\right) + (\varphi_N(x) - \Phi(\beta))\, r_N^+\left(\tfrac{x}{N}\right) \right\}.$$

By using the fact that $\varphi_N(x)$ is solution of the traffic equation (2.16) the first three lines in last display are equal to 0. It remains to see that the last line in last display is also equal to 0. For that purpose we observe that we can rewrite it as

$$\frac{\kappa}{N^\theta} \sum_{\xi\in\Omega_N} f(\xi)\tilde{\nu}_{\varphi_N(\cdot)}(\xi) \left\{ \sum_{x\in\Lambda} [\varphi_N(x) + \varphi_N(N-x) - (\Phi(\alpha) + \Phi(\beta))]\, r_N^-\left(\tfrac{x}{N}\right) \right\}.$$

Hence it is sufficient to show that

$$\varphi_N(x) + \varphi_N(N-x) = \Phi(\alpha) + \Phi(\beta). \tag{3.2}$$

To see this it is enough to sum (2.16) evaluated at x and the same equation evaluated at $N-x$, and use the fact that

$$\sum_{y\in\Lambda_N} p(y-(N-x)) = \sum_{y\in\Lambda_N} p(y-x), \quad r^+\left(\tfrac{N-x}{N}\right) = r_N^-\left(\tfrac{x}{N}\right), \quad r^-\left(\tfrac{N-x}{N}\right) = r_N^+\left(\tfrac{x}{N}\right)$$

and

$$\sum_{y\in\Lambda_N} p(y-(N-x))\varphi_N(y) = \sum_{y\in\Lambda_N} p(y-x)\varphi_N(N-y).$$

From this we get that

$$(\varphi_N(x) + \varphi_N(N-x)) \left(\sum_{y\in\Lambda_N} p(y-x) + \kappa N^{-\theta}\left(r_N^+\left(\tfrac{x}{N}\right) + r_N^-\left(\tfrac{x}{N}\right)\right)\right)$$
$$= \sum_{y\in\Lambda_N} p(y-x)\,(\varphi_N(y) + \varphi_N(N-y)) + \kappa N^{-\theta}\,(\Phi(\beta) + \Phi(\alpha))\left(r_N^+\left(\tfrac{x}{N}\right) + r_N^-\left(\tfrac{x}{N}\right)\right).$$

We have seen above that the matrix $D_N - P_N$ is invertible, so that this discrete equation with unknown $\psi(\cdot) = \varphi_N(\cdot) + \varphi_N(N-\cdot))$ has a unique solution. Since the constant function $\psi(\cdot) = \Phi(\beta) + \Phi(\alpha)$ is a solution, we can conclude that $\varphi_N(x) + \varphi_N(N-x) = \Phi(\alpha) + \Phi(\beta)$ and this ends the proof. □

Proof of Proposition 2.5 Note that for each $x \in \Lambda_N$ we have

$$E_{\nu_{ss}^N}[g(\xi(x))] = \int_{\Omega_N} g(\xi(x)) d\nu_{ss}^N(\xi) = \sum_{k=0}^{\infty} \frac{g(k)(\varphi_N(x))^k}{Z(\varphi_N(x))g(k)!}$$
$$= \frac{\varphi_N(x)}{Z(\varphi_N(x))} \sum_{k=0}^{\infty} \frac{(\varphi_N(x))^k}{g(k)!} = \varphi_N(x).$$

On the other hand, since μ_{ss}^N is a stationary measure, by writing for each x that $E_{\mu_{ss}^N}[\mathcal{L}_N f_x] = 0$ (recall (2.13)) with $f_x(\eta) = \eta(x)$, we get directly that $(\Phi(\alpha) + \Phi(\beta)) E_{\mu_{ss}^N}[\eta(x)]$ is the solution of the traffic equation (2.16). So, by uniqueness of the solution of (2.16) we conclude that $E_{\mu_{ss}^N}[\eta(x)] = \varphi_N(x)$.

□

4 Proof of Theorem 2.15

Proof For $x \in \Lambda_N$ and $\varepsilon > 0$ we define[3] the box of length $2\varepsilon N + 1$ centered around x

$$I_{\varepsilon N}(x) = [x - \varepsilon N, x + \varepsilon N] \cap \Lambda_N$$

and for a function $h : \Lambda_N \to \mathbb{R}$ we define its average in this box by

$$A[h, I_{\varepsilon N}(x)] = \frac{1}{\# I_{\varepsilon N}(x)} \sum_{y \in I_{\varepsilon N}(x)} h(y) \tag{4.1}$$

where # is the counting measure. Fix $\varepsilon > 0$. By adding and subtracting the term

$$\frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} G(\tfrac{x}{N}) F\left(A[\varphi_N, I_{\varepsilon N}(x)], \tfrac{x}{N}\right)$$

and using the triangle inequality, we get that

$$\begin{aligned}
&\left|\frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) F\left(\varphi_N(x), \tfrac{x}{N}\right) - \int_0^1 G(u) F\left([\Phi(\alpha) + \Phi(\beta)]\, \bar{\rho}(u), u\right) du\right| \\
&\quad \leq \left|\frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) \left(F\left(\varphi_N(x), \tfrac{x}{N}\right) - F\left(A[\varphi_N, I_{\varepsilon N}(x)], \tfrac{x}{N}\right)\right)\right| \\
&\quad + \left|\frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) F\left(A[\varphi_N, I_{\varepsilon N}(x)], \tfrac{x}{N}\right) - F\left([\Phi(\alpha) + \Phi(\beta)]\, \bar{\rho}\left(\tfrac{x}{N}\right), \tfrac{x}{N}\right)\right| \\
&\quad + \left|\frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) F\left([\Phi(\alpha) + \Phi(\beta)]\, \bar{\rho}\left(\tfrac{x}{N}\right), \tfrac{x}{N}\right)\right. \\
&\quad \left. - \int_0^1 G(u) F\left([\Phi(\alpha) + \Phi(\beta)]\, \bar{\rho}(u), u\right) du\right|.
\end{aligned}$$

Note that the last sum is a Riemann sum, and since $\bar{\rho}$ is a continuous function, see Lemma A.2, the last line in last display vanishes as N goes to ∞. So, it is enough to prove that the limit as $N \to +\infty$ and then $\varepsilon \to 0$ of the remaining terms vanishes. By the triangle inequality,

[3] We assume for simplicity that εN is an integer.

(2.18) and by using the fact F is Lipschitz in the first component we have that

$$\begin{aligned}
&\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\left(F\left(\varphi_N(x),\tfrac{x}{N}\right)-F\left(A[\varphi_N,I_{\varepsilon N}(x)],\tfrac{x}{N}\right)\right)\right|\\
&\quad+\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\left\{F\left(A[\varphi_N,I_{\varepsilon N}(x)],\tfrac{x}{N}\right)-F\left([\Phi(\alpha)+\Phi(\beta)]\bar{\rho}\left(\tfrac{x}{N}\right),\tfrac{x}{N}\right)\right\}\right|\\
&\lesssim \frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}\left|G\left(\tfrac{x}{N}\right)\right|\left|E_{\mu_{ss}^N}[\eta(x)-A[\eta,I_{\varepsilon N}(x)]]\right|\\
&\quad+\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\left\{F\left(A[\varphi_N,I_{\varepsilon N}(x)],\tfrac{x}{N}\right)-F\left([\Phi(\alpha)+\Phi(\beta)]\bar{\rho}\left(\tfrac{x}{N}\right),\tfrac{x}{N}\right)\right\}\right|\\
&=(\mathrm{I})+(\mathrm{II}),
\end{aligned}$$

where above we have used the notation $f(t)\lesssim g(t)$ to express the fact there exists a constant C independent of t such that $f(t)\le Cg(t)$, for every t.

We show separately that (I) and (II) go to zero in order to conclude the proof. For (I), by the stationary property of μ_{ss}^N and Fubini's Theorem, we obtain that

$$\left|E_{\mu_{ss}^N}[\eta(x)-A[\eta,I_{\varepsilon N}(x)]]\right|\le \mathbb{E}_{\mu_{ss}^N}\left[\left|\int_0^1 \eta_t(x)-A[\eta_t,I_{\varepsilon N}(x)]dt\right|\right]. \tag{4.2}$$

By Lemmas 5.3, 5.4 and 5.5 in [5] (taking $\mu_N=\mu_{ss}^N$) we obtain that

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\mathbb{E}_{\mu_{ss}^N}\left[\left|\int_0^1 \eta_t(x)-A[\eta_t,I_{\varepsilon N}(x)]dt\right|\right]=0,$$

which means that, we can replace the occupation number at site x by its average in a box of length $\#I_{\varepsilon N}(x)$. Hence (I) goes to zero. For (II), we introduce the notation $\|G\|_{1,N}=\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}|G\left(\tfrac{x}{N}\right)|$, and use the triangle inequality and the fact F is Lipschitz in the first component, to get the bound

$$\begin{aligned}
(\mathrm{II})&\lesssim\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\left\{F\left(A[\varphi_N,I_{\varepsilon N}(x)],\tfrac{x}{N}\right)-F\left(\tfrac{\Phi(\alpha)+\Phi(\beta)}{\|G\|_{1,N}}\int_0^1|G(u)|\bar{\rho}(u)du,\tfrac{x}{N}\right)\right\}\right|\\
&\quad+\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\left\{F\left(\tfrac{\Phi(\alpha)+\Phi(\beta)}{\|G\|_{1,N}}\int_0^1|G(u)|\bar{\rho}(u)du,\tfrac{x}{N}\right)-F\left([\Phi(\alpha)+\Phi(\beta)]\bar{\rho}\left(\tfrac{x}{N}\right),\tfrac{x}{N}\right)\right\}\right|\\
&\lesssim\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}|G\left(\tfrac{x}{N}\right)|\left|A[\varphi_N,I_{\varepsilon N}(x)]-\tfrac{\Phi(\alpha)+\Phi(\beta)}{\|G\|_{1,N}}\int_0^1|G(u)|\bar{\rho}(u)du\right|\\
&\quad+\frac{\Phi(\alpha)+\Phi(\beta)}{\#\Lambda_N}\sum_{x\in\Lambda_N}|G\left(\tfrac{x}{N}\right)|\left|\tfrac{1}{\|G\|_{1,N}}\int_0^1|G(u)|\bar{\rho}(u)du-\bar{\rho}\left(\tfrac{x}{N}\right)\right|\\
&\lesssim\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}|G\left(\tfrac{x}{N}\right)|A[\varphi_N,I_{\varepsilon N}(x)]-[\Phi(\alpha)+\Phi(\beta)]\int_0^1|G(u)|\bar{\rho}(u)du\right|\\
&\quad+[\Phi(\alpha)+\Phi(\beta)]\left|\int_0^1|G(u)|\bar{\rho}(u)du-\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}|G\left(\tfrac{x}{N}\right)|\bar{\rho}\left(\tfrac{x}{N}\right)\right|.
\end{aligned}$$

Recalling that $A[\varphi_N, I_{\varepsilon N}(x)] = \frac{1}{\# I_{\varepsilon N}(x)} \sum_{y \in I_{\varepsilon N}(x)} \varphi_N(y)$ and $\varphi_N(y) = [\Phi(\alpha) + \Phi(\beta)] E_{\mu_{ss}^N}[\eta(y)]$, from Theorem A.1 and Lemma A.2, we conclude that (II) goes to zero.

It remains to prove that $\bar{\rho}(\cdot)$ takes values in $[\tilde{\alpha}, \tilde{\beta}]$. By Proposition 2.3 and Proposition 2.5 we have that for any $x \in \Lambda_N$,

$$E_{\mu_{ss}^N}[\eta(x)] \in [\tilde{\alpha}, \tilde{\beta}].$$

By using (4.2) and Theorem A.1 we conclude[4] that for any continuous positive function $G : [0, 1] \to \mathbb{R}$ we have

$$\tilde{\alpha} \int_0^1 G(u)du \le \int_0^1 G(u)\bar{\rho}(u)du \le \tilde{\beta} \int_0^1 G(u)du.$$

Then, for any $v \in [0, 1]$, we choose a sequence $(G_k)_k$ of positive continuous functions defined on $[0, 1]$ and converging in the distributional sense to the Dirac mass on v. Applying the last inequality to G_k and letting k going to infinity, we conclude, since $\bar{\rho}$ is continuous that $\tilde{\alpha} \le \bar{\rho}(v) \le \tilde{\beta}$. □

5 Proof of Theorem 2.20

To prove this theorem we apply Corollary 4.5.27 of [54] and therefore we need to check the following facts:

1. For all $C^0([0, 1])$, denoting

$$\Lambda_N(G) = \log E_{\nu_{ss}^N}\left[e^{N\langle \pi, G\rangle}\right], \tag{5.1}$$

   we have that $\lim_{N\to\infty} \frac{\Lambda_N(G)}{N}$ exists, it is equal to $\Lambda(G)$ and it is finite.
2. The functional Λ is Gateaux differentiable.
3. $\Lambda : C^0([0, 1]) \to \mathbb{R}$ is lower semi-continuous.
4. The sequence $\{\pi_N\}_{N\ge 1}$ is exponentially tight.

Let us prove these four items. The first one is the content of Proposition 5.1. For the second one, we recall that the partition function Z is analytic on $(0, \infty)$. Then, for all $G, H \in C^0([0, 1])$ we have that

$$\lim_{t\to 0} \frac{\Lambda(G + tH) - \Lambda(G)}{t} = \lim_{t\to 0} \int_0^1 \frac{\log\left(\frac{Z(e^{(G+tH)(u)}\Phi(\bar{m}(u)))}{Z(e^{G(u)}\Phi(\bar{m}(u)))}\right)}{t} du,$$

and by dominated convergence theorem we get that

$$\begin{aligned} \lim_{t\to 0} \frac{\Lambda(G + tH) - \Lambda(G)}{t} &= \int_0^1 \lim_{t\to 0} \frac{\log\left(\frac{Z(e^{(G+tH)(u)}\Phi(\bar{m}(u)))}{Z(e^{G(u)}\Phi(\bar{m}(u)))}\right)}{t} du \\ &= \int_0^1 R\left(e^{G(u)}\bar{\rho}(u)\right) H(u)du, \end{aligned} \tag{5.2}$$

[4] Theorem A.1 is established for continuous functions and we need to apply it to the non continuous function $\iota_u^{\varepsilon}(\cdot) = (2\varepsilon)^{-1} \mathbb{1}_{|\cdot - u| \le \varepsilon}$, $u \in [0, 1]$. This can be done by a standard approximation argument.

so that the second item is proved. The third item is trivial since Λ is a continuous function. Indeed, by using the fact that $\bar{\rho}(\cdot)$ is a bounded function and $\log Z$ is continuously differentiable, we have that for any $G, H \in C^0([0,1])$,

$$|\Lambda(H) - \Lambda(G)| \leq \int_0^1 \left| \log\left(Z\left(e^{H(u)}\bar{\rho}(u)\right)\right) - \log\left(Z\left(e^{G(u)}\bar{\rho}(u)\right)\right)\right| du \leq C\|H-G\|_\infty$$

where $C = \sup_{\varphi\in[0,c]} \left| \frac{d}{d\varphi} \log\left(Z(\varphi)\right)\right|$ with $c = \|\bar{\rho}\|_\infty e^{\sup\{\|G\|_\infty, \|H\|_\infty\}}$. Therefore if $\lim_{H\to G} \Lambda(H) = \Lambda(G)$ in $C^0([0,1])$. Hence Λ is continuous and therefore lower semi-continuous. It remains to prove the forth item. We recall that $\mathcal{K}_A = \{\mu \in \mathcal{M}\ ;\ |\mu([0,1])| \leq A\}$ is a compact subset of $\mathcal{M}$ for the weak-$\star$ topology. Hence to prove the exponential tightness of the sequence $\left\{\pi^N\right\}_{N\geq 1}$ it is sufficient to prove that

$$\limsup_{A\to\infty} \limsup_{N\to\infty} \frac{1}{\#\Lambda_N} \log\ P_{\nu_{ss}^N}\left[\frac{1}{N}\sum_{x\in\Lambda_N} \xi(x) \geq A\right] = -\infty. \tag{5.3}$$

By Markov's inequality we have that

$$P_{\nu_{ss}^N}\left[\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N}\xi(x)\geq A\right] \leq e^{-NA+\frac{\Lambda_N(\mathbf{1})}{N}}$$

where Λ_N is defined in (5.1) and $\mathbf{1}$ is the constant function on $[0,1]$ equal to 1. Since $\frac{\Lambda_N(1)}{N} \to \Lambda(\mathbf{1})$ (see Proposition 5.1) we get the result.

It remains to prove Proposition 5.1. Recall that Λ_N is defined by (5.1).

Proposition 5.1 *For any continuous function $G : [0,1] \to \mathbb{R}$ we have that*

$$\lim_{N\to\infty}\left|\frac{\Lambda_N(G)}{N} - \Lambda(G)\right| = 0.$$

Proof We denote for $\lambda \in \mathbb{R}$ and $x \in \Lambda_N$ the exponential moment of $\xi(x)$: $M_N(\lambda, x) = E_{\nu_{ss}^N}\left[e^{\lambda\xi(x)}\right]$. We have that

$$M_N(\lambda, x) = \frac{1}{Z(\varphi_N(x))}\sum_{k=0}^{\infty}\frac{\left(e^\lambda\varphi_N(x)\right)^k}{g(k)!} = \frac{Z\left(e^\lambda\varphi_N(x)\right)}{Z(\varphi_N(x))}.$$

Observe that since ν_{ss}^N is product we have that

$$\begin{aligned}\frac{\Lambda_N(G)}{N} &= \frac{1}{N}\log E_{\nu_{ss}^N}\left[e^{N\langle\pi, G\rangle}\right] = \frac{1}{N}\log E_{\nu_{ss}^N}\left[e^{\sum_{x\in\Lambda_N} G\left(\frac{x}{N}\right)\xi_x}\right] = \frac{1}{N}\log E_{\nu_{ss}^N}\left[\prod_{x\in\Lambda_N} e^{G\left(\frac{x}{n}\right)\xi_x}\right]\\ &= \frac{1}{N}\sum_{x\in\Lambda_N}\log M_N\left(G(\tfrac{x}{N}), x\right) = \frac{1}{N}\sum_{x\in\Lambda_N}\log\frac{Z\left(e^{G\left(\frac{x}{n}\right)}\varphi_N(x)\right)}{Z(\varphi_N(x))}\\ &= \frac{1}{N}\sum_{x\in\Lambda_N}\mathfrak{F}\left(\varphi_N(x), G(\tfrac{x}{N})\right)\end{aligned}$$

where $\mathfrak{F}(\varphi, u) = \log\frac{Z\left(e^{G(u)}\varphi\right)}{Z(\varphi)}$. Observe that $\mathfrak{F}$ is Lipschitz in the first component because $Z : [0,\infty) \to [1,\infty)$ is analytic. Hence we can apply Theorem 2.15 and obtain the result. □

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Appendix A: Macroscopic Properties of NESS in Boundary Driven Long Range Exclusion

In this section, we present some results for the stationary behaviour of the open boundary exclusion process with long jumps based on previous results from [5, 10, 13]. Recall (2.12) and note that we consider the process speeded up in the time scale $\Theta(N)$ as in (2.11). Recall that $\tilde{\alpha}, \tilde{\beta} \in (0, 1)$.

A.1. Hydrostatic Limit

Let $\mathcal{M}^+$, be the space of positive measures on $[0, 1]$ with total mass bounded by 1 and equipped with the weak $\star$-topology. For any $\eta \in \Omega^N$ the empirical measure $\pi_{ex}^N := \pi_{ex}^N(\eta) \in \mathcal{M}^+$ is defined by

$$\pi_{ex}^N(\eta) = \frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} \eta_x \delta_{\frac{x}{N}}(du). \tag{A.1}$$

Let P^N be the probability measure on $\mathcal{M}^+$ obtained as the pushforward of μ_{ss}^N by π_{ex}^N. We denote the action of $\pi_{ex}^N \in \mathcal{M}^+$ on a continuous function $G : [0, 1] \to \mathbb{R}$ by

$$\langle \pi_{ex}^N, G \rangle = \int_{[0,1]} G(u)\, \pi_{ex}^N(du).$$

Theorem A.1 *(Hydrostatic limit in mean) For any $G \in C^0([0, 1])$ we have*

$$\lim_{N \to \infty} E_{\mu_{ss}^N} \left[\langle \pi_{ex}^N, G \rangle - \int_0^1 \bar{\rho}(u) G(u) du \right] = 0,$$

where

(a) *for $\theta < 0$ and $\gamma \in (0, 2)$, $\bar{\rho}(u) = \dfrac{V_0(u)}{V_1(u)}$.*
(b) *for $\theta = 0$ and $\gamma \in (0, 2) - \{1\}$, $\bar{\rho}(\cdot)$ is the unique weak solution of (2.23), with $\hat{\kappa} = \kappa$.*
(c) *for $\theta \in (0, \gamma - 1)$ and $\gamma \in (1, 2)$, $\bar{\rho}(\cdot)$ is the unique weak solution of (2.24).*
(d) *for $\theta = \gamma - 1$ and $\gamma \in (1, 2)$, $\bar{\rho}(\cdot)$ is the weak solution of (2.25) with $\hat{\kappa} = \kappa d$.*
(e) *for $\theta > 0$ and $\gamma \in (0, 1]$ or for $\theta > \gamma - 1$ and $\gamma \in (1, 2)$, $\bar{\rho}(\cdot)$ is the weak solution of (2.27) with $\hat{M} = \frac{\tilde{\alpha}+\tilde{\beta}}{2}$, i.e. $\bar{\rho} = \frac{\tilde{\alpha}+\tilde{\beta}}{2}$.*

Proof We start by noting that a simple computation based on the fact that the mass of the system is finite, shows that the sequence $\{P^N\}_{N \geq 2}$ is tight and that all limit points are concentrated on measures $\pi(du)$ which are absolutely continuous with respect to Lebesgue measure on $[0, 1]$, i.e. $\pi(du) = \rho(u)du$. Let us also introduce $\{\bar{\pi}^N\}_{N \geq 2} = \left\{ E_{\mu_{ss}^N} \left[\pi_{ex}^N \right] \right\}_{N \geq 2}$ which forms a sequentially compact sequence of $\mathcal{M}^+$ whose limit points $\bar{\pi}(du)$ are absolutely continuous with respect to the Lebesgue measure on $[0, 1]$, i.e. $\bar{\pi}(du) = \bar{\rho}(u)du$. Let

$\bar{\pi}(du) = \bar{\rho}(u)du$ be a limit point of $\{\bar{\pi}^N\}_{N\geq 2}$. Without loss of generality we can consider a subsequence for which $\{P^N\}_{N\geq 2}$ is also converging to a limit point denoted by P^* (the corresponding expectation is denoted by E^*). To lighten notation, in the sequel, we assume that we are taking the limit according to this subsequence, even if it is not specified. Observe that $\bar{\rho} = E^*[\rho]$. Our goal is to show that $\bar{\rho}$ is unique and given as in Theorem A.1.

If $\gamma \in (0, 2)$, the energy estimates of Sect. 3.3 of [10] show that for $\theta \geq 0$

$$E^*\left[\|\rho\|^2_{\gamma/2}\right] < \infty$$

and for $\theta \leq 0$

$$E^*\left[\int_0^1 \left\{\frac{(\tilde{\alpha} - \rho(u))^2}{u^\gamma} + \frac{(\tilde{\beta} - \rho(u))^2}{(1-u)^\gamma}\right\} du\right] < \infty.$$

From Jensen's inequality, we have also

$$\|\bar{\rho}\|_{\gamma/2} < \infty, \quad \int_0^1 \left\{\frac{(\tilde{\alpha} - \bar{\rho}(u))^2}{u^\gamma} + \frac{(\tilde{\beta} - \bar{\rho}(u))^2}{(1-u)^\gamma}\right\} du < \infty.$$

It remains now to check that $\bar{\rho}$ satisfies the other conditions in the notions of stationary weak solutions.

Recall (2.11). Note that

$$\begin{aligned}\Theta(N)\mathcal{L}_N\left\langle\pi^N_{ex}, G\right\rangle &= \frac{\Theta(N)}{\#\Lambda_N}\sum_{x\in\Lambda_N}(\mathbb{L}_N G)\left(\tfrac{x}{N}\right)\eta(x)\\ &+ \frac{\kappa\Theta(N)}{N^\theta \#\Lambda_N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\left\{r_N^-\left(\tfrac{x}{N}\right)(\tilde{\alpha} - \eta(x)) + r_N^+\left(\tfrac{x}{N}\right)\left(\tilde{\beta} - \eta(x)\right)\right\}\end{aligned} \tag{A.2}$$

where the action of $\mathbb{L}_N$ on functions G is defined by

$$(\mathbb{L}_N G)\left(\tfrac{x}{N}\right) = \sum_{y\in\Lambda_N} p(y-x)\left[G\left(\tfrac{y}{N}\right) - G\left(\tfrac{x}{N}\right)\right]. \tag{A.3}$$

Taking the expectation with respect to μ^N_{ss} on (A.2), we get, from stationarity, that

$$\begin{aligned}0 &= \frac{\Theta(N)}{\#\Lambda_N}\sum_{x\in\Lambda_N}(\mathbb{L}_N G)\left(\tfrac{x}{N}\right)E_{\mu^N_{ss}}[\eta(x)]\\ &+ \frac{\kappa\Theta(N)}{N^\theta\#\Lambda_N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\left\{r_N^-\left(\tfrac{x}{N}\right)\left(\tilde{\alpha} - E_{\mu^N_{ss}}[\eta(x)]\right) + r_N^+\left(\tfrac{x}{N}\right)\left(\tilde{\beta} - E_{\mu^N_{ss}}[\eta(x)]\right)\right\}.\end{aligned} \tag{A.4}$$

Recall (2.9). We define the functions $r_N^\pm : [0, 1] \to \mathbb{R}$ as the linear interpolation of $r_N^-\left(\tfrac{x}{N}\right)$ and $r_N^+\left(\tfrac{x}{N}\right)$ for all $x \in \Lambda_N$ with $r_N^\pm(0) = r_N^\pm\left(\tfrac{1}{N}\right)$ and $r_N^\pm(1) = r_N^\pm\left(\tfrac{N-1}{N}\right)$. By Lemma 3.3 in [13], for $0 < \gamma < 2$ we have that

$$\lim_{N\to\infty} N^\gamma\left(r_N^-\right)(u) = r^-(u), \quad \lim_{N\to\infty} N^\gamma\left(r_N^+\right)(u) = r^+(u) \tag{A.5}$$

uniformly in $[a, 1-a]$ for $a \in (0, 1)$ and from that lemma it also follows that

$$\lim_{N\to\infty} N^\gamma(\mathbb{L}_N G)(u) = (\mathbb{L}G)(u) \tag{A.6}$$

uniformly in $[a, 1-a]$, for all functions G with compact support included in $[a, 1-a]$.

Now, we split the analysis by taking into account the value of θ.

Case $\theta < 0$: In this regime we take $G \in C_c^\infty((0,1))$ and $\Theta(N) = N^{\theta+\gamma}$. Observe that (A.4) can be written as

$$0 = \langle \bar{\pi}^N, N^{\gamma+\theta}\mathbb{L}_N G\rangle - \kappa \langle \bar{\pi}^N, G N^\gamma (r_N^- + r_N^+)\rangle \\ + \frac{\kappa}{\#\Lambda_N} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) N^\gamma \left\{ r_N^-\left(\tfrac{x}{N}\right)\tilde{\alpha} + r_N^+\left(\tfrac{x}{N}\right)\tilde{\beta} \right\} \tag{A.7}$$

From the weak convergence together with (A.5) and (A.6), last display converges, as $N \to +\infty$, to

$$-\kappa \int_0^1 G(u)\left\{\bar{\rho}(u)V_1(u)du - V_0(u)\right\}du = 0$$

which implies that $\bar{\rho}(u) = \frac{V_0(u)}{V_1(u)}$ for almost every u (see also Remark (2.14) in [10]).

Case $\theta = 0$: In this regime we take $G \in C_c^\infty((0,1))$ and $\Theta(N) = N^\gamma$. From weak convergence together with (A.5) and (A.6) we get that

$$0 = \int_0^1 (\mathbb{L}G)(u)\bar{\rho}(u)du - \kappa \int_0^1 G(u)\bar{\rho}(u)V_1(u)du + \kappa \int_0^1 G(u)V_0(u)du.$$

Hence we have that $F_{RD}(\bar{\rho}, G) = 0$ for any $G \in C_c^\infty((0,1))$ (with $\hat{\kappa} = \kappa$).

Case $\theta \in (0, \gamma - 1)$ **and** $\gamma \in (1, 2)$: In this regime we take $G \in C_c^\infty((0,1))$ and $\Theta(N) = N^\gamma$. From the weak convergence, the rightmost term in the first line of (A.4) converges, as $N \to +\infty$, to

$$\int_0^1 (\mathbb{L}G)(u)\bar{\rho}(u)du.$$

Moreover, the term on the second line of (A.2) can be bounded from above by a constant times

$$N^{\gamma-\theta-1} \sum_{x\in\Lambda_N} x^{-\gamma} G\left(\tfrac{x}{N}\right) \lesssim N^{-\theta}$$

plus lower order terms in N which vanish as $N \to +\infty$. Hence we have that $F_{Dir}(\bar{\rho}, G) = 0$ for any $G \in C_c^\infty((0,1))$.

Case $\theta = \gamma - 1$ **and** $\gamma \in (1, 2)$: In this regime we take $G \in C^\infty([0,1])$ and $\Theta(N) = N^\gamma$. We start by noting that from Lemma 5.1 of [5], we have that

$$\lim_{N\to\infty} N^{-1} \sum_{x\in\Lambda_N} \left| N^\gamma (\mathbb{L}_N G)\left(\tfrac{x}{N}\right) - (\mathbb{L}G)\left(\tfrac{x}{N}\right)\right| = 0$$

for functions $G \in C^\infty([0,1])$ and $\gamma \in (0,2)$. Moreover, the first term on the second line of (A.4) can be rewritten as

$$\frac{\kappa N^\gamma}{N^{\gamma-1}\#\Lambda_N}\left(\tilde{\alpha} - E_{\mu_{ss}^N}\Big[A[\eta, I_{\varepsilon N}(0)]\Big]\right) \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) r_N^-\left(\tfrac{x}{N}\right) \\ + \frac{\kappa N^\gamma}{N^{\gamma-1}\#\Lambda_N} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) r_N^-\left(\tfrac{x}{N}\right)\left(E_{\mu_{ss}^N}\Big[A[\eta, I_{\varepsilon N}(0)]\Big] - \eta(x)\right) \tag{A.8}$$

plus analogous terms with respect to the right boundary. The expectation of the second term above converges to zero as $N \to +\infty$ from Lemma 5.9 of [5] (see Remark 5.10 there). We note that, to apply Lemma 5.9 of [5], which is written with the time integral and in the L^1 sense, in last argument, we need to use the fact that μ_{ss}^N is a stationary measure, to introduce the time integral in the rightmost term of (A.8) and then use the aforementioned lemma. Now, for the first term, we can perform a Taylor expansion on G obtaining the following expression,

$$\frac{\kappa N^{\gamma} G(0)\left(\tilde{\alpha}-E_{\mu_{ss}^N}\left[A[\eta, I_{\varepsilon N}(0)]\right]\right)}{N^{\gamma-1}\,\#\Lambda_N}\sum_{x\in\Lambda_N} r_N^-\left(\tfrac{x}{N}\right) + \frac{\kappa G'(0)\left(\tilde{\alpha}-E_{\mu_{ss}^N}\left[A[\eta, I_{\varepsilon N}(0)]\right]\right)}{\#\Lambda_N}\sum_{x\in\Lambda_N} x r_N^-\left(\tfrac{x}{N}\right)$$

plus lower order terms in N. Observe that

$$\frac{\kappa\left|G'(0)\left(\tilde{\alpha}-E_{\mu_{ss}^N}\left[A[\eta, I_{\varepsilon N}(0)]\right]\right)\right|}{\#\Lambda_N}\sum_{x\in\Lambda_N} x r_N^-\left(\tfrac{x}{N}\right) \lesssim \frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N} x^{1-\gamma} = \mathcal{O}(N^{1-\gamma}),$$

which goes to zero as N goes to infinity. Finally, the remaining term can be treated by using the weak convergence, the fact that the limiting measure $\bar{\pi}(du)$ is absolutely continuous with respect to the Lebesgue measure with density $\bar{\rho}$ and also that

$$\lim_{N\to+\infty}\sum_{x\in\Lambda_N} r_N^{\pm}\left(\tfrac{x}{N}\right) = d.$$

Hence we get that $F_{Rob}(\bar{\rho}, G) = 0$ (with $\hat{\kappa} = \kappa d$) for any $G \in C^\infty([0,1])$.

Case $\theta > \gamma - 1$ **and** $\gamma \in (1,2)$: The proof in this regime is completely analogous to the previous case. Nevertheless, we could also obtain the result as a consequence of the hydrostatic limit, with a convergence in probability, by following the same strategy described in [51]. Since we do not need this stronger convergence to attain our results, we did not pursue this issue here.

Case $\theta > 0$ **and** $\gamma \in (0,1)$: In this case the analysis of the first term of (A.4) is given by an approximation argument of the operator $\mathbb{L}$ given in Lemma 5.1 of [5]. Since $G(x/N)$ and $\eta(x)$ are bounded we know that the second term of (A.4) is bounded by a constant times

$$N^{\gamma-\theta-1}\sum_{x\in\Lambda_N}\left\{r_N^-\left(\tfrac{x}{N}\right)+r_N^+\left(\tfrac{x}{N}\right)\right\}.$$

Since $\gamma \in (0,1)$, we lose the convergence of the partial sum above. However, it is not difficult to see that

$$N^{\gamma-\theta-1}\sum_{x\in\Lambda_N}\left\{r_N^-\left(\tfrac{x}{N}\right)+r_N^+\left(\tfrac{x}{N}\right)\right\} \lesssim N^{-\theta}, \tag{A.9}$$

which vanishes as N goes to ∞. Hence $F_{Neu}(\bar{\rho}, G) = 0$ for any $G \in C^\infty([0,1])$.

In the two last cases, we have also to show that $\int_0^1 \bar{\rho}(u)du = \frac{\tilde{\alpha}+\tilde{\beta}}{2}$. In fact, since in these cases, $F_{Neu}(\bar{\rho}, G) = 0$ for any $G \in C^\infty([0,1])$, we can conclude that $\bar{\rho}$ is equal to a constant M by using the same argument as in Lemma 2.14 when showing that $\|\bar{\rho}\|_{\gamma/2} = 0$.

Recall the definition of $\bar{\pi}^N$ given at the beginning of the proof. We consider a sequence $(\chi^\varepsilon)_{0<\varepsilon<1/4}$ of smooth functions with values in [0, 1], symmetric with respect to 1/2, equal to 0 on $[0, \varepsilon/2]$ and to 1 on $[\varepsilon, 1/2]$. Observe that $(\chi^\varepsilon)_\varepsilon$ converges in L^1 as ε goes to 0 to the constant function equal to 1. Taking $G = \chi^\varepsilon$ in (A.4) we get

$$\frac{1}{N}\sum_{x\in\Lambda_N}\chi^\varepsilon\left(\tfrac{x}{N}\right)\left\{\tilde{\alpha}N^\gamma r_N^-\left(\tfrac{x}{N}\right)+\tilde{\beta}N^\gamma r_N^+\left(\tfrac{x}{N}\right)\right\}$$
$$=\int_0^1\chi^\varepsilon(u)(N^\gamma r_N^-(u)+N^\gamma r_N^+(u))\bar{\pi}^N(du)+N^{-\theta}\frac{1}{N}\sum_{x\in\Lambda_N}(N^\gamma\mathbb{L}_N\chi^\varepsilon)\left(\tfrac{x}{N}\right)E_{\mu_{ss}^N}[\eta(x)]\,.$$

By (A.6) and (A.5), we can then replace in the previous expression $N^\gamma\mathbb{L}_N$ by $\mathbb{L}$ and $N^\gamma r_N^\pm$ by $r^\pm$. Recall that $\bar{\pi}^N(du)$ converges weakly to $\bar{\rho}(u)du = Mdu$. Since $\theta > 0$ we get

$$\int_0^1\chi^\varepsilon(u)V_1(u)du = M\int_0^1\chi^\varepsilon(u)V_0(u)du. \tag{A.10}$$

In the case $\gamma\in(0,1)$ the functions V_0 and V_1 are integrable and we get by sending ε to 0 that

$$M=\frac{\int_0^1 V_1(u)du}{\int_0^1 V_0(u)du}=\frac{\tilde{\alpha}+\tilde{\beta}}{2}.$$

In the case $\gamma\in(1,2)$ the integrals are diverging but

$$\int_0^1\chi^\varepsilon(u)V_1(u)du\sim\frac{c_\gamma}{\gamma(\gamma-1)}\varepsilon^{1-\gamma}\left[\tilde{\alpha}+\tilde{\beta}\right],\quad\int_0^1\chi^\varepsilon(u)V_0(u)du\sim 2\frac{c_\gamma}{\gamma(\gamma-1)}\varepsilon^{1-\gamma}$$

therefore we get again that

$$M=\frac{\tilde{\alpha}+\tilde{\beta}}{2}.$$

Conclusion: All limit points of the sequence $\{\bar{\pi}^N(du)\}_{N\geq 2}$ are in the form $\bar{\rho}(u)du$ where $\bar{\rho}$ is a weak solution of the hydrostatic equation. By uniqueness of weak solutions for these equations, $\bar{\rho}$ is unique and therefore the sequence is converging, without extracting a subsequence, to this unique weak solution. □

Lemma A.2 *The profiles $\bar{\rho}$ in Theorem A.1 are continuous in* $(0, 1)$.

Proof In the case $\theta < 0$ the claim follows easily since the profile is explicit. We consider now $\theta \geq 0$. If $\gamma \in (1, 2)$, by definition of a weak solution, we know that $\bar{\rho}$ is bounded and belongs to $\mathcal{H}^{\gamma/2}$ and from Theorem 8.2 of [24], we conclude that $\bar{\rho}$ is $\frac{\gamma-1}{2}$-Hölder in $[0, 1]$, therefore continuous in $(0, 1)$. If $\gamma\in(0, 1)$ and $\theta > 0$, the profile is constant and therefore continuous. The only missing case is $\gamma \in (0, 1)$ and $\theta = 0$. We have hence to prove that the stationary solution ρ of the regional fractional reaction–diffusion equation (2.23) is continuous in $(0, 1)$ when $\gamma \in (0, 1)$. It is known that if $\gamma\in(0, 1]$, the condition $f \in \mathcal{H}^{\gamma/2}$ does not guarantee, contrarily to the case $\gamma \in (1, 2)$, that f is continuous. Therefore the continuity property of ρ can only result from the fact that ρ satisfies the weak formulation of (2.23). This property is a consequence of potential theory for (fractional) Shrödinger theory developed in [17], more exactly of Proposition 6.1 of that article that we restate in our particular context. Before doing so, we introduce a few notations.

The fractional Laplacian $|\Delta|^{\gamma/2}$ on $\mathbb{R}$ is the operator acting on functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$\int_{\mathbb{R}}\frac{|f(u)|}{(1+|u|)^{1+\gamma}}du<\infty \tag{A.11}$$

as

$$-\left(|\Delta|^{\gamma/2} f\right)(u) = c_\gamma \lim_{\epsilon\to 0} \int_{\mathbb{R}} \mathbb{1}_{|u-v|\geq\epsilon} \frac{f(v)-f(u)}{|u-v|^{1+\gamma}} dv, \tag{A.12}$$

for any $u \in \mathbb{R}$ if the limit exists (which is for example the case for smooth compactly supported functions). It can be extended into the weak fractional Laplacian (that we denote abusively by the same notation) by duality: For any f satisfying (A.11), $|\Delta|^{\gamma/2} f$ is the distribution (or generalized function) on $\mathbb{R}$ satisfying the identity $\langle |\Delta|^{\gamma/2} f, h\rangle = \langle f, |\Delta|^{\gamma/2} h\rangle$, for any compactly supported function h on $\mathbb{R}$.

Property 6.1 of [17] claims that if $f : \mathbb{R} \to \mathbb{R}$ is a solution, in the distributional sense[5], on $(0, 1)$ of the equation

$$|\Delta|^{\gamma/2} f + qf = 0 \tag{A.13}$$

then f is continuous on $(0, 1)$ as soon as the function $q : (0, 1) \to \mathbb{R}$ belongs to the (local) Kato class of exponent γ, i.e.

$$\lim_{r\to 0} \sup_{u\in\mathbb{R}} \int_{u-r}^{u+r} \frac{(\mathbb{1}_{[\varepsilon,1-\varepsilon]} q)(u)}{|u-v|^{1-\gamma}} dv = 0$$

for any $\varepsilon \in (0, 1/2)$. It is not difficult to see that if ρ is the weak solution of (2.23) and is extended by 0 outside of $(0, 1)$, then ρ satisfies, in the distributional sense, (A.13) on $(0, 1)$ for q given as a linear combination of r^- and r^+ (defined by (2.22)). Hence to conclude the proof of the lemma, it is sufficient to prove that $r^\pm$ belong to the (local) Kato class of exponent γ. This exercise is left to the interested reader. □

A.2. Fractional Fick's Law of the Boundary Driven Exclusion

By adapting the strategy of [13] we can obtain the "fractional Fick's Law" which is given in the next theorem. The expression of the current is given by

$$\begin{aligned} W_x^{ex}(\eta) = &\sum_{\substack{1\leq y\leq x-1\\ x-1<z\leq N-1}} p(z-y)(\eta(y)-\eta(z)) \\ &+ \frac{\kappa}{N^\theta}\Bigg[\sum_{x\leq z\leq N-1} r_N^-\left(\tfrac{z}{N}\right)(\tilde{\alpha}-\eta(z)) - \sum_{1\leq y\leq x-1} r_N^+\left(\tfrac{y}{N}\right)(\tilde{\beta}-\eta(y))\Bigg]. \end{aligned} \tag{A.14}$$

Theorem A.3 (Fractional Fick's law) *Let $\bar{\rho}(\cdot)$ be the hydrostatic profile of the boundary driven exclusion given in Theorem A.1. For $u \in (0, 1)$ the following fractional Fick's law holds, apart from the case $\theta = 0$ and $\gamma = 1$:*

$$\lim_{N\to\infty} \frac{1}{B_N(\theta)} E_{\mu_{ss}^N}\left[W_{[uN]}^{ex}\right] = \int_0^1 h_\theta(u)\bar{\rho}(u)du + C(\tilde{\alpha}, \tilde{\beta}, \theta), \tag{A.15}$$

where

$$B_N(\theta) := N^{1-\gamma}\mathbb{1}_{\theta\geq 0} + N^{1-\theta-\gamma}\mathbb{1}_{\theta<0}, \tag{A.16}$$

[5] It means that for any $h \in C_c^\infty((0, 1))$, $\int_0^1 f(u)|\Delta|^{\gamma/2}h(u)\,du + \int_0^1 f(u)q(u)\,du = 0$.

the function $h_\theta : (0,1) \to \mathbb{R}$ *is given by*

$$h_\theta(u) = \begin{cases} c_\gamma \left(\frac{\kappa}{\gamma} \mathbf{1}_{\theta \le 0} + \frac{1}{1-\gamma} \mathbf{1}_{\theta \ge 0} \right) [(1-u)^{1-\gamma} - u^{1-\gamma}], & \text{if } \gamma \neq 1, \\ c_\gamma \mathbf{1}_{\theta \ge 0} [\log(1-u) - \log(u)], & \text{if } \gamma = 1 \end{cases} \tag{A.17}$$

and

$$C(\tilde{\alpha}, \tilde{\beta}, \theta) = \frac{c_\gamma \kappa (\tilde{\alpha} - \tilde{\beta})}{\gamma(2-\gamma)} \mathbf{1}_{\theta \le 0}, \tag{A.18}$$

This implies that

(a) *for* $\theta < 0$,

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N^{1-\theta-\gamma}} \mathbb{E}_{\mu_{ss}^N} \left[W_{[uN]}^{ex} \right] &= \kappa \int_u^1 (\tilde{\alpha} - \bar{\rho}(v)) r^-(v) dv - \kappa \int_0^u (\tilde{\beta} - \bar{\rho}(v)) r^+(v) dv \\ &= \kappa c_\gamma \gamma^{-1} \int_0^1 \frac{\tilde{\alpha} - \tilde{\beta}}{v^\gamma + (1-v)^\gamma} dv; \end{aligned} \tag{A.19}$$

(b) *for* $\theta = 0$,

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N^{1-\gamma}} \mathbb{E}_{\mu_{ss}^N} \left[W_{[uN]}^{ex} \right] = c_\gamma \int_0^u \int_u^1 \frac{\bar{\rho}(v) - \bar{\rho}(w)}{(w-v)^{1+\gamma}} dw dv + \kappa \int_u^1 (\tilde{\alpha} - \bar{\rho}(v)) r^-(v) dv \\ - \kappa \int_0^u (\tilde{\beta} - \bar{\rho}(v)) r^+(v) dv; \end{aligned} \tag{A.20}$$

(c) *for* $\theta > 0$,

$$\lim_{N \to \infty} \frac{1}{N^{1-\gamma}} \mathbb{E}_{\mu_{ss}^N} \left[W_{[uN]}^{ex} \right] = c_\gamma \int_0^u \int_u^1 \frac{\bar{\rho}(v) - \bar{\rho}(w)}{(w-v)^{1+\gamma}} dw dv. \tag{A.21}$$

Proof Since the measure μ_{ss}^N is stationary and $\mathcal{L}_N \eta_x = W_x^{ex}(\eta) - W_{x+1}^{ex}(\eta)$ for all $x \in \Lambda_N$ and for all η, it follows that $E_{\mu_{ss}^N}[W_x^{ex}] = E_{\mu_{ss}^N}[W_1^{ex}]$, for all $x \in \Lambda_N$. Then we can write

$$\begin{aligned} E_{\mu_{ss}^N}[W_1^{ex}] = &\frac{\kappa N^{-\theta}}{\#\Lambda_N} \sum_{z \in \Lambda_N} z \left[\tilde{\alpha} - E_{\mu_{ss}^N}[\eta(z)] \right] \sum_{y \le 0} p(z-y) \\ &+ \frac{\kappa N^{-\theta}}{\#\Lambda_N} \sum_{y \in \Lambda_N} (N-1-y) \left[E_{\mu_{ss}^N}[\eta(y)] - \tilde{\beta} \right] \sum_{z \ge N} p(z-y) \\ &+ \frac{1}{\#\Lambda_N} \sum_{z \in \Lambda_N} \sum_{y=1}^{z-1} p(z-y)(z-y) \left[E_{\mu_{ss}^N}[\eta(y)] - E_{\mu_{ss}^N}[\eta(z)] \right]. \end{aligned} \tag{A.22}$$

We define the linear interpolation functions $\tilde{r}_N^\pm : [0,1] \to \mathbb{R}$, such that for all $z \in \Lambda_N$ we have that

$$\tilde{r}_N^- \left(\tfrac{z}{N} \right) = \sum_{y \ge z} y p(y) \quad \text{and} \quad \tilde{r}_N^+ \left(\tfrac{z}{N} \right) = - \sum_{y \le z-N} y p(y). \tag{A.23}$$

Using (A.22) it is not difficult to see that

$$\frac{1}{B_N(\theta)} E_{\mu_{ss}^N}[W_1^{ex}] = \frac{1}{\#\Lambda_N} \sum_{x \in \Lambda_N} h_\theta^N \left(\tfrac{x}{N} \right) E_{\mu_{ss}^N}[\eta(x)] + \frac{C_\theta^N}{B_N(\theta)}, \tag{A.24}$$

where

$$h_\theta^N\left(\tfrac{x}{N}\right) = \frac{\kappa N^{1-\theta}}{B_N(\theta)}\left[-\tfrac{x}{N} r_N^-\left(\tfrac{x}{N}\right) + \left(\tfrac{N-1-x}{N}\right) r_N^+\left(\tfrac{x}{N}\right)\right] + \frac{\tilde{r}_N^-\left(\tfrac{x}{N}\right) - \tilde{r}_N^+\left(\tfrac{x}{N}\right)}{B_N(\theta)},$$

and

$$C_\theta^N := \frac{\kappa}{N^\theta \,\#\Lambda_N}\left[\tilde{\alpha}\sum_{z\in\Lambda_N}\sum_{y\leq 0} z p(z-y) - \tilde{\beta}\sum_{y\in\Lambda_N}\sum_{z\geq N}(N-1-y)p(z-y)\right].$$

From (A.23), we get the following convergence

$$\lim_{N\to\infty}\left|h_\theta^N\left(\tfrac{[uN]}{N}\right) - h_\theta(u)\right| = 0 \tag{A.25}$$

which holds uniformly for $u \in (a, 1-a)$ (with $0 < a < 1$ fixed), and the function $h_\theta : (0,1) \to \mathbb{R}$ is given in (A.17). We note that h_θ is singular at $u = 0$ and $u = 1$ if $\gamma \in (1,2)$, but it is integrable in $[0,1]$ for $\gamma \in (0,2)$. Moreover, it is easy to see that, for $u \in (0,1)$ and for $\gamma \neq 1$ we have that

$$|h_\theta(u)| \lesssim u^{1-\gamma} + (1-u)^{1-\gamma}. \tag{A.26}$$

Regarding the last term of (A.24), a simple computation shows that

$$\lim_{N\to\infty}\frac{1}{B_N(\theta)}C_\theta^N = C(\tilde{\alpha},\tilde{\beta},\theta),$$

where $C(\tilde{\alpha},\tilde{\beta},\theta)$ was defined in (A.18). Now, note that from (A.25) and the fact that $|\eta(x)| \leq 1$, we get that

$$\lim_{N\to\infty}\left|\frac{1}{\#\Lambda_N}\sum_{x\in NI(a)}\left(h_\theta^N\left(\tfrac{x}{N}\right) - h_\theta\left(\tfrac{x}{N}\right)\right)E_{\mu_{ss}^N}[\eta(x)]\right| = 0, \tag{A.27}$$

where $I(a) = [a, 1-a]$ and $NI(a) = [Na, N(1-a)] \cap \mathbb{N}$. From (A.26) and the fact that $|\eta(x)| \leq 1$ we get that

$$\left|\frac{1}{\#\Lambda_N}\sum_{x\in(NI(a))^c} h_\theta^N\left(\tfrac{x}{N}\right)E_{\mu_{ss}^N}[\eta(x)]\right| \lesssim \left[a^{2-\gamma} + (1-a)^{2-\gamma}\right].$$

Moreover, from Theorem A.1 and, for h_θ^a a continuous extension of the function h_θ restricted to $I(a)$, we get that

$$\lim_{N\to\infty}\left|\frac{1}{\#\Lambda_N}\sum_{x\in\Lambda_N} h_\theta^a\left(\tfrac{x}{N}\right)E_{\mu_{ss}^N}[\eta(x)] - \int_0^1 h_\theta^a(u)\bar{\rho}(u)du\right| = 0. \tag{A.28}$$

Now, from (A.24) and (A.28) sending first $N \to \infty$ and then $a \to 0$ we obtain (A.15).

The other expressions of the limiting current given at the end of the theorem are achieved by using properties of the integrals and the fact that the limit does not depend on the variable u. To check them properly, let us consider for instance $\theta > 0$ and $\gamma \neq 1$. Since the limit does not depend on u, we have that

$$c_\gamma\int_0^u\int_u^1\frac{\bar{\rho}(v)-\bar{\rho}(w)}{(w-v)^{1+\gamma}}dwdv = c_\gamma\int_0^1\int_0^u\int_u^1\frac{\bar{\rho}(v)-\bar{\rho}(w)}{(w-v)^{\gamma+1}}dwdvdu.$$

Using Fubini's theorem twice, last display equals to

$$c_\gamma \int_0^1 \int_v^1 \int_v^w \frac{\bar{\rho}(v) - \bar{\rho}(w)}{(w-v)^{\gamma+1}} du dw dv = c_\gamma \int_0^1 \int_v^1 \frac{\bar{\rho}(v) - \bar{\rho}(w)}{(w-v)^{\gamma}} dw dv.$$

Finally, a simple computation, based again on Fubini's theorem, shows that last display is equal to

$$\int_0^1 h_\theta(v)\bar{\rho}(v)dv.$$

This ends the case $\theta > 0$. The cases $\theta \le 0$ can be obtained by performing similar computations to the ones above, plus the fact that

$$\kappa \int_0^1 \int_u^1 \tilde{\alpha} r^-(v) dv\, du - \kappa \int_0^1 \int_0^u \tilde{\beta} r^+(v) dv\, du = \frac{c_\gamma \kappa (\tilde{\alpha} - \tilde{\beta})}{\gamma(2-\gamma)}.$$

Finally, we note that the second equality in item a) is obtained by algebraic manipulations using the fact that $\bar{\rho}(u) = \dfrac{V_0(u)}{V_1(u)}$. □

References

1. Bahadoran, C.: A quasi-potential for conservation laws with boundary conditions. arXiv Preprint arXiv:1010.3624 (2010)
2. Barré, J., Bernardin, C., Chétrite, R.: Density large deviations for multidimensional stochastic hyperbolic conservation laws. J. Stat. Phys. **170**(3), 466–491 (2018)
3. Bellettini, G., Bertini, L., Mariani, M., Novaga, M.: Γ-entropy cost for scalar conservation laws. Arch. Ration. Mech. Anal. **195**, 261–309 (2010)
4. Bernardin, C.: Stationary nonequilibrium properties for a heat conduction model. Phys. Rev. E **78**, 021134 (2008)
5. Bernardin, C., Cardoso, P., Goncalves, P., Scotta, S.: Hydrodynamic limit for a boundary driven super-diffusive symmetric exclusion. arXiv Preprint arXiv:2007.01621 (2020)
6. Bernardin, C., Gonçalves, P., Jara, M.: 3/4-Fractional superdiffusion in a system of harmonic oscillators perturbed by a conservative noise. Arch. Ration. Mech. Anal. **220**(2), 505–542 (2016)
7. Bernardin, C., Gonçalves, P., Jara, M., Sasada, M., Simon, M.: From normal diffusion to superdiffusion of energy in the evanescent flip noise limit. J. Stat. Phys. **159**(6), 1327–1368 (2015)
8. Bernardin, C., Gonçalves, P., Jara, M., Simon, M.: Interpolation process between standard diffusion and fractional diffusion. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **54**(3), 1731–1757 (2018)
9. Bernardin, C., Gonçalves, P., Jara, M., Simon, M.: Nonlinear perturbation of a noisy Hamiltonian lattice field model: universality persistence. Commun. Math. Phys. **361**(2), 605–659 (2018)
10. Bernardin, C., Gonçalves, P., Jiménez-Oviedo, B.: A microscopic model for a one parameter class of fractional Laplacians with Dirichlet boundary conditions. Arch. Ration. Mech. Anal. **239**(1), 1–48 (2020)
11. Bernardin, C., Gonçalves, P., Jiménez-Oviedo, B.: Slow to fast infinitely extended reservoirs for the symmetric exclusion process with long jumps. Markov Process. Relat. Fields **25**, 217–274 (2019)
12. Bernardin, C., Gonçalves, P., Sethuraman, S.: Occupation times of long-range exclusion and connections to KPZ class exponents. Probab. Theory Relat. Fields **166**(1), 365–428 (2016)
13. Bernardin, C., Jiménez-Oviedo, B.: Fractional Fick's law for the boundary driven exclusion process with long jumps. ALEA **14**(1), 473–501 (2017)
14. Bernardin, C., Kannan, V., Lebowitz, J.L., Lukkarinen, J.: Harmonic systems with bulk noises. J. Stat. Phys. **146**(4), 800–831 (2012)
15. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Macroscopic fluctuation theory. Rev. Mod. Phys. **87**(2), 593 (2015)
16. Bodineau, T., Derrida, B.: Phase fluctuations in the ABC model. J. Stat. Phys. **145**, 745–762 (2011)
17. Bogdan, K., Byczkowski, T.: Potential theory of Schrödinger operator based on fractional Laplacian. Probab. Math. Stat. **20**(2), 293–335 (2000)

18. Cane, G.: Superdiffusion transition for a noisy harmonic chain subject to a magnetic field. arXiv Preprint arXiv:2201.03373 (2022)
19. Cardoso, P., Gonçalves, P., Jiménez-Oviedo, B.: Hydrodynamic behavior of long-range symmetric exclusion with a slow barrier: diffusive regime to appear in AIHP Sec. B. arXiv Preprint arXiv:2111.02868 (2021)
20. Chebloun, P., Grosskinsky, S.: Condensation in stochastic particle systems with stationary product measures. J. Stat. Phys. **154**(1–2), 432–465 (2014)
21. Das, S.G., Dhar, A., Saito, K., Mendl, C.B., Spohn, H.: Numerical test of hydrodynamic fluctuation theory in the Fermi-Pasta-Ulam chain. Phys. Rev. E **90**(1), 012124 (2014)
22. De Masi, A., Ferrari, P.A.: A remark on the hydrodynamics of the zero range process. J. Stat. Phys. **36**, 81–87 (1984)
23. Derrida, B.: Non-equilibrium steady states: fluctuations and large deviations of the density and of the current. J. Stat. Mech. **2007**, P07023 (2007)
24. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012)
25. Evans, M.R.: Phase transitions in one-dimensional nonequilibrium systems. Braz. J. Phys. **30**(1), 42–57 (2000)
26. Evans, M.R., Hanney, T.: Nonequilibrium statistical mechanics of the zero-range process and related models. J. Phys. A **38**(19), R195 (2005)
27. Gonçalves, P.: Hydrodynamics for symmetric exclusion in contact with reservoirs. In: Stochastic Dynamics Out of Equilibrium, Institut Henri Poincaré, Paris, France, 2017, Springer Proceedings in Mathematics and Statistics book series, pp. 137–205 (2019)
28. Gonçalves, P., Jara, M.: Density fluctuations for exclusion processes with long jumps. Probab. Theory Relat. Fields **170**(1), 311–362 (2018)
29. Gonçalves, P., Scotta, S.: Diffusive to super-diffusive behavior in boundary driven exclusion. Markov Process. Relat. Fields **28**, 149–178 (2022)
30. Guan, Q.-Y., Ma, Z.-M.: Reflected symmetric α-stable processes and regional fractional Laplacian. Probab. Theory Relat. Fields **134**(4), 649–694 (2006)
31. Hurtado, P.I., Garrido, P.L.: A violation of universality in anomalous Fourier's law. Sci. Rep. **6**(1), 1–10 (2016)
32. Jara, M.: Current and density fluctuations for interacting particle systems with anomalous diffusive behavior. eprint at arXiv:0901.0229 (2009)
33. Jara, M., Komorowski, T., Olla, S.: Superdiffusion of energy in a chain of harmonic oscillators with noise. Commun. Math. Phys. **339**(2), 407–453 (2015)
34. Jensen, L.: The asymmetric exclusion process in one dimension. Ph.D. Dissertation, New York University, New York (2000)
35. Kipnis, C.: Central limit theorems for infinite series of queues and applications to simple exclusion. Ann. Probab. **14**(2), 397–408 (1986)
36. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Springer-Verlag, New York (1999)
37. Kundu, A., Bernardin, C., Saito, K., Kundu, A., Dhar, A.: Fractional equation description of an open anomalous heat conduction set-up. J. Stat. Mech.: Theory Exp. **2019**(1), 013205 (2019)
38. Lepri, S., Politi, A.: Density profiles in open superdiffusive systems. Phys. Rev. E **83**, 030107(R) (2011)
39. Levine, E., Mukamel, D., Schütz, G.M.: Zero-range process with open boundaries. J. Stat. Phys. **120**(5–6), 759–778 (2005)
40. Liggett, T.M.: Interacting Particle Systems. Classics in Mathematics. Springer-Verlag, Berlin (2005)
41. Mariani, M.: Large deviations principles for stochastic scalar conservation laws. Probab. Theory Relat. Fields **147**(3–4), 607–648 (2010)
42. Popkov, V., Schadschneider, A., Schmidt, J., Schütz, G.: Fibonacci family of dynamical universality classes. PNAS **11241**, 12645–12650 (2015)
43. Popkov, V., Schmidt, J., Schütz, G.M.: Universality classes in two-component driven diffusive systems. J. Stat. Phys. **160**(4), 835–860 (2015)
44. Saito, K., Sasada, M., Suda, H.: 5/6-Superdiffusion of energy for coupled charged harmonic oscillators in a magnetic field. Commun. Math. Phys. **372**(1), 151–182 (2019)
45. Sethuraman, S.: On microscopic derivation of a fractional stochastic Burgers equation. Commun. Math. Phys. **341**(2), 625–665 (2016)
46. Spitzer, F.: Interaction of Markov processes. Adv. Math. **5**(2), 246–290 (1970)
47. Sethuraman, S., Shahar, D.: Hydrodynamic limits for long-range asymmetric interacting particle systems. Electron. J. Probab. **23**, 1–54 (2018)
48. Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer-Verlag, Berlin (1991)

49. Spohn, H.: Nonlinear fluctuating hydrodynamics for anharmonic chains. J. Stat. Phys. **154**(5), 1191–1227 (2014)
50. Spohn, H., Stolz, G.: Nonlinear fluctuating hydrodynamics in one dimension: the case of two conserved fields. J. Stat. Phys. **160**, 861–884 (2015)
51. Tsunoda, K.: Hydrostatic limit for exclusion process with slow boundary revisited. RIMS Kôkyûroku Bessatsu **B79**, 149–162 (2020)
52. Vanicat, M.: Exact solution to integrable open multi-species SSEP and macroscopic fluctuation theory. J. Stat. Phys. **166**, 1129–1150 (2017)
53. Varadhan, S.R.S.: Large deviations for the asymmetric simple exclusion process. Stochastic analysis on large scale interacting systems. Adv. Stud. Pure Math. **39**, 1–27 (2004)
54. Zeitouni, A.D.O., Dembo, O.: Large Deviations Techniques and Applications. Springer-Verlag, New York (1998)