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ORIGINAL PAPER

Some aspects on the computational implementation of diverse terms arising in mixed virtual element formulations

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Abstract
In the present paper, we describe the computational implementation of some integral terms that arise from mixed virtual element methods (mixed-VEM) in two-dimensional pseudostress-velocity formulations. The implementation presented here considers any polynomial degree $k \geq 0$ in a natural way by building several local matrices of small size through the matrix multiplication and the Kronecker product. In particular, we apply the foregoing mentioned matrices to the Navier-Stokes equations with Dirichlet boundary conditions, whose mixed-VEM formulation was originally proposed and analyzed in a recent work using virtual element subspaces for $H(\mathrm{div})$ and H^1, simultaneously. In addition, an algorithm is proposed for the assembly of the associated global linear system for Newton's iteration. Finally, we present a numerical example in order to illustrate the performance of the mixed-VEM scheme and confirm the expected theoretical convergence rates.

Keywords Mixed virtual element method · High-order approximations · Computational implementation · Navier-Stokes problem · Pseudostress-velocity formulation · Augmented formulation

1 Introduction

The virtual element method (VEM) was introduced in [1] for the Poisson equation, is one of the high-order discretization schemes for the approximation of solutions to partial differential equations, that can be seen as a generalization of the standard

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finite element method. In other words, the method extends the classical finite element technique to general polygonal and polyhedral meshes. Moreover, according to [2], additional advantages of VEM schemes, when compared with finite volume methods, mimetic finite difference method, and related techniques, are given by its solid mathematical ground, the simplicity of the respective computational coding, and the quality of the numerical results provided. On the other hand, regarding to purely mixed virtual element techniques, that is based on dual-mixed variational formulations, the method was initially developed in [3], and more recently extended in [4–7], and [8]. In particular, edge and face VEM spaces in 2D and 3D, which together with the nodal and volume spaces constitute a discrete complex, were developed in [4], whereas [5] generalizes the results of [4] to the case of variable coefficients. In turn, [6] and [8] provide the first analysis of a virtual element method for a mixed variational formulation of the Stokes and Navier-Stokes problems, respectively, in which the pseudostress and the velocity are the only unknowns, whereas the pressure is computed via a postprocessing formula. Additionally, the analysis presented in [8] allows to study problems of the same nature as Navier-Stokes, such as the Boussinesq problem, where a mixed virtual element method was introduced and analyzed in [9]. For several other contributions on VEM and mixed-VEM we refer for instance to [10–14], and [15]. In particular, in [14], a virtual element method for the quasilinear equation was presented and analysed.

The previously mentioned references omit to present aspects related to computational implementation of mixed-VEM schemes. For this reason, the main goal of this paper is to describe a computational approach for mixed-VEM methods in order to obtain high-order approximations, without imposing a certain programming language. More precisely, the reason for this contribution is that there is no much literature that explains how to program the subspaces of virtual elements accurately (particularly the mixed ones). Up to the authors' knowledge, some works such as [1–4, 13, 16, 17] and [18] explain specific aspects of the computational implementation, but in general, they do not describe the structures employed. Up to the authors' knowledge, in [19], the authors present the first paper concerning a detailed implementation of virtual element method. However, this contribution focus only on the lowest order approximations for Poison equation. Next, in [20], the authors describe in Spanish some specific aspects on the computational implementation of the a mixed-VEM method for the 2D linear Brinkman model proposed and analyzed in [7]. In fact, the latter extends the approach used in [7], in order to obtain implementation techniques for several mixed-VEM schemes, including those with nonlinearities (see, e.g., [8]).

The paper is organized as follows. In Section 2, we introduce the virtual element subspaces for $H(\mathrm{div})$-conforming and H^1-conforming that will be employed. This includes the main ingredients for the polygonal mesh structure, the definitions of the local degrees of freedom, and the projections to be employed, along with a description about the explicit calculation of each projector. Next, in Sections 3 and 4, we present the main contributions of this work. Indeed, in Section 3, we describe the assembling of the local matrices associated with the projectors respect to the local virtual spaces, whereas in Section 4, the computational aspects required for the construction of some local terms arising in mixed-primal virtual element formulations are described. In addition, we remark in advance that each discrete operator is built for an arbitrary polynomial degree $k \geq 0$, which means that we will develop a high-order computational approach. Finally, in Section 5, in order to illustrate the use of the matrices introduced in previous sections, we recall the boundary value problem and its mixed-VEM formulation introduced and analyzed in [8]. More precisely, we present a mixed virtual element method for the two-dimensional pseudostress-velocity formulation of the Navier-Stokes equations with Dirichlet boundary conditions. Therein, the continuous and discrete formulations are presented. Furthermore, we propose an algorithm for the assembly of the associated global linear system for Newton's iteration, and then, a numerical example illustrating the performance of the mixed-VEM scheme and confirming the expected theoretical convergence rates is presented.

1.1 Notations

We end the present section by providing some notations to be used along the paper. Indeed, in what follows, we consider a bounded domain $\Omega \subseteq \mathbb{R}^2$ with boundary Γ. Moreover, standard terminology for Lebesgue and Sobolev spaces will be adopted where, given a generic scalar functional space H, we denote by $\mathbf{H}$ and $\mathbb{H}$ be the corresponding vectorial and tensorial counterparts, respectively. For example, given an integer $\ell \geq 0$ and $U \subseteq \mathbb{R}^2$, we let $\mathrm{P}_\ell(U)$ be the space of polynomials on U of degree up to ℓ, whereas $\mathbf{P}_\ell(U)$ stands for its vectorial version, that is, $\mathbf{P}_\ell(U) := [\mathrm{P}_\ell(U)]^2$. In addition, $\mathbb{P}_\ell(U) := [\mathrm{P}_\ell(U)]^{2\times 2}$ corresponds to its tensorial version.

Now, we employ $\mathbf{v} \otimes \mathbf{w}$ to stand the usual dyadic product for two column vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, that is, $\mathbf{v} \otimes \mathbf{w} := \mathbf{v}\,\mathbf{w}^{\mathrm{t}} \in \mathbb{R}^{2\times 2}$. On the other hand, given two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$, we denote the matrix concatenation of $\mathbf{A}$ and $\mathbf{B}$ as follows:

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \end{array}\right] \quad \text{if } m = p\,, \qquad \text{or} \qquad \left[\begin{array}{c} \mathbf{A} \\ \hdashline \mathbf{B} \end{array}\right] \quad \text{if } n = q\,.$$

In addition, we let $\mathrm{kron}(\cdot,\cdot)$ be the usual Kronecker product, that is:

$$\mathrm{kron}(\mathbf{A},\mathbf{B}) := \left[\begin{array}{c:c:c} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \hdashline \vdots & \ddots & \vdots \\ \hdashline a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{array}\right] \in \mathbb{R}^{(mp)\times(nq)}\,.$$

Finally, when we write $\mathbf{A} := [a_{ij}] \in \mathbb{R}^{m\times n}$, it means that $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

2 The virtual element subspaces

In this section, we recall two local element subspaces, usually used in the design of virtual element schemes. More precisely, we introduce the spaces $\mathbf{H}_k^K$ (see (2.3)) and V_k^K (see (2.14)) for $H(\mathrm{div})$-conforming and H^1-conforming elements, respectively. In order to do that, we let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω in polygonal elements, where h denotes the largest of its diameters. For theoretical purposes, it is assumed that $\mathcal{T}_h$ satisfies some conditions described at the beginning of [8, Section 3.1].

In what follows, we consider K an arbitrary element of $\mathcal{T}_h$. It is important to realize that the decomposition $\mathcal{T}_h$ needs a quite more sophisticated computational structure than those used in classical finite element methods. Indeed, we recall here that $\mathcal{T}_h$ can contain elements with several shapes that affect important aspects, such as the following: the number of edges and the calculation of its diameter. In particular, any structure that is implemented for mesh management, from a connectivity point of view, must be able to indicate:

- the number of nodes (points)
- the number of edges
- the number of elements
- the number of boundary edges
- for any element:
    - the number of nodes of the element
    - the global index (identifier) of a node of the element
    - the local indexes of the nodes in a specific edge
    - the global indexes of the nodes in a specific edge
    - the global index of an edge of the element
- for any edge:
    - the global index of a node of the edge
    - the global index of the edge
    - the global index of the neighbor element shared by the edge
    - the global indexes of the elements that contain the edge
    - the orientation in a specific elements that contain the edge
    - an identifier that establishes if the edge is in the boundary

Here, the orientation of an edge corresponds to a boolean identifier that indicates the ordering (independent of the element that contains it) of its extreme nodes. In addition, in the case of geometric aspects, for all $K \in \mathcal{T}_h$, it must be able to calculate:

- the number of vertices, or equivalently, the number of edges
- the coordinates of a specific vertex
- the coordinates of the barycenter of K
- the area of K
- the diameter of K
- the midpoints, normal vectors, and lengths of each of its edges

2.1 H(div; *K*)-conforming subspace, associated bilinear form, and projection

Let e be an edge of $\mathcal{T}_h$ with midpoint x_e and length h_e. Thus, given an integer $\ell \geq 0$, we consider the following set of $k+1$ normalized monomials on e:

$$\mathcal{B}_\ell(e) \;:=\; \left\{ \left(\frac{x - x_e}{h_e} \right)^j \right\}_{0 \leq j \leq \ell}, \tag{2.1}$$

which constitutes a basis of $\mathrm{P}_\ell(e)$. Similarly, given an element $K \in \mathcal{T}_h$ with barycenter $\mathbf{x}_K$ and diameter h_K, we define the following set of $\frac{1}{2}(\ell+1)(\ell+2)$ normalized monomials on K:

$$\mathcal{B}_\ell(K) \;:=\; \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^{\boldsymbol{\alpha}} \right\}_{0 \leq |\boldsymbol{\alpha}| \leq \ell}, \tag{2.2}$$

which is a basis of $\mathrm{P}_\ell(K)$. It is important to remark that in (2.2) we use the multi-index notation, where given $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \mathbb{R}^2$ and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)^{\mathrm{t}}$, with nonnegative integers α_1, α_2, we let $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2$.

Next, we introduce the auxiliary local virtual element space of order $k \geq 0$ (see, e.g., [4, 5])

$$\begin{aligned} \mathbf{H}_k^K \;:=\; \Big\{ \tau := (\tau_1, \tau_2)^{\mathrm{t}} \in \mathbf{H}(\mathrm{div}; K) \cap \mathbf{H}(\mathrm{rot}; K) \;:\quad & \tau \cdot \boldsymbol{n}|_e \in \mathrm{P}_k(e) \\ \forall \text{ edge } e \in \partial K\,, \quad \mathrm{div}(\tau) \in \mathrm{P}_k(K)\,, \;\; \text{and} \;\; & \mathrm{rot}(\tau) \in \mathrm{P}_{k-1}(K) \Big\}, \end{aligned} \tag{2.3}$$

where $\mathrm{rot}(\tau) := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}$ and $\mathrm{P}_{-1}(K) := \{0\}$. Moreover, the local degrees of freedom for $\tau \in \mathbf{H}_k^K$ are given by (see [4, 5])

$$\begin{aligned} m^H_{q,\boldsymbol{n}}(\tau) &:= \int_e \boldsymbol{\tau} \cdot \boldsymbol{n}\, q && \forall\, q \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \in \partial K\,, \\ m^H_{q,\mathrm{div}}(\tau) &:= \int_K \tau \cdot \nabla q && \forall\, q \in \mathcal{B}_k(K) \setminus \{1\}\,, \\ m^H_{\mathbf{q},\mathrm{rot}}(\tau) &:= \int_K \tau \cdot \mathbf{q} && \forall\, \mathbf{q} \in \mathcal{G}_k^{\perp}(K)\,, \end{aligned} \tag{2.4}$$

where $\mathcal{G}_k^{\perp}(K)$ is a basis of $(\nabla \mathrm{P}_{k+1}(K))^{\perp} \cap \mathbf{P}_k(K)$, which corresponds to the $\mathbf{L}^2(K)$-orthogonal of $\nabla \mathrm{P}_{k+1}(K)$ in $\mathbf{P}_k(K)$. Then, according to the cardinalities of $\mathcal{B}_k(e)$ and $\mathcal{B}_k(K)$, and the dimensions of $\mathbf{P}_k(K)$ and $\nabla \mathrm{P}_{k+1}(K)$, it follows that the cardinality of $\mathcal{G}_k^{\perp}(K)$ is $\frac{1}{2}k(k+1)$. Thus, the number of local degrees of freedom defined in (2.4) (i.e., the dimension of $\mathbf{H}_k^K$) is given by:

$$\begin{aligned} n_k^H \;=\; n_k^H(K) &:= (k+1)\, d_K \;+\; \left\{ \frac{(k+1)(k+2)}{2} - 1 \right\} \;+\; \frac{k(k+1)}{2} \\ &= (k+1)(d_K + k + 1) - 1\,, \end{aligned} \tag{2.5}$$

where d_K corresponds to the number of edges in K. Furthermore, it was proved in [4, Section 3.4] that, for every $K \in \mathcal{T}_h$, these n_k^H local degrees of freedom are unisolvent in $\mathbf{H}_k^K$.

On the other hand, we employ the space $\mathbf{H}_k^K$ to define a tensor virtual element space $\mathbb{H}_k^K$ as:

$$\mathbb{H}_k^K \;=\; \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};K) \;:\quad (\tau_{i1},\,\tau_{i2})^{\mathtt{t}} \in \mathbf{H}_k^K \quad \forall\, i \in \{1,2\}\Big\}. \tag{2.6}$$

In other words, $\mathbb{H}_k^K$ is a subspace of $\mathbb{H}(\mathbf{div};K)$, where each row of $\boldsymbol{\tau} \in \mathbb{H}_k^K$ belongs to $\mathbf{H}_k^K$. According to this, it is natural to consider for each $\boldsymbol{\tau} \in \mathbb{H}_k^K$, the following $2n_k^H$ local degrees of freedom:

$$\begin{aligned}
\boldsymbol{m}^H_{\mathbf{q},\boldsymbol{n}}(\boldsymbol{\tau}) &:= \int_e \boldsymbol{\tau}\boldsymbol{n}\cdot\mathbf{q} \qquad && \forall\,\mathbf{q} \in \boldsymbol{\mathcal{B}}_k(e)\,, \quad \forall \text{ edge } e \in \partial K\,,\\
\boldsymbol{m}^H_{\mathbf{q},\mathbf{div}}(\boldsymbol{\tau}) &:= \int_K \boldsymbol{\tau} : \nabla\mathbf{q} && \forall\,\mathbf{q} \in \boldsymbol{\mathcal{B}}_k(K)\setminus\{(1,0)^{\mathtt{t}},(0,1)^{\mathtt{t}}\}\,,\\
\boldsymbol{m}^H_{\boldsymbol{\rho},\mathbf{rot}}(\boldsymbol{\tau}) &:= \int_K \boldsymbol{\tau} : \boldsymbol{\rho} && \forall\,\boldsymbol{\rho} \in \boldsymbol{\mathcal{G}}_k^{\perp}(K)\,,
\end{aligned} \tag{2.7}$$

where

$$\boldsymbol{\mathcal{B}}_\ell(e) := \big\{(q,0)^{\mathtt{t}} \,:\, q \in \mathcal{B}_\ell(e)\big\} \cup \big\{(0,q)^{\mathtt{t}} \,:\, q \in \mathcal{B}_\ell(e)\big\}\,,$$

$$\boldsymbol{\mathcal{B}}_\ell(K) := \big\{(q,0)^{\mathtt{t}} \,:\, q \in \mathcal{B}_\ell(K)\big\} \cup \big\{(0,q)^{\mathtt{t}} \,:\, q \in \mathcal{B}_\ell(K)\big\}\,,$$

and

$$\boldsymbol{\mathcal{G}}_k^{\perp}(K) \;:=\; \left\{\begin{pmatrix}\mathbf{q}\\ \mathbf{0}\end{pmatrix} \,:\, \mathbf{q} \in \mathcal{G}_k^{\perp}(K)\right\} \cup \left\{\begin{pmatrix}\mathbf{0}\\ \mathbf{q}\end{pmatrix} \,:\, \mathbf{q} \in \mathcal{G}_k^{\perp}(K)\right\}.$$

Furthermore, according to the degrees of freedom defined in (2.4) and (2.7), it is possible to define a bilinear form $\mathcal{S}_H^K : \mathbb{H}_k^K \times \mathbb{H}_k^K \to \mathbb{R}$ based on these. Indeed, let $K \in \mathcal{T}_h$ and consider the union of all local degrees of freedom (cf. (2.4)) of a given $\tau \in \mathbf{H}^1(K)$, in a set $\{m^H_{i,K}(\tau)\}_{i=1}^{n_k^H}$. Then, as usual, let $\{\Psi_j^K\}_{j=1}^{n_k^H}$ be the canonical basis of $\mathbf{H}_k^K$, that is, given $i = 1, 2, \ldots, n_k^H$, Ψ_i^K is the unique element in $\mathbf{H}_k^K$ such that:

$$m^H_{j,K}(\Psi_i^K) \;=\; \delta_{ij} \qquad \forall\, j = 1,2,\ldots,n_k^H\,, \tag{2.8}$$

where, in particular, there holds:

$$\tau \;=\; \sum_{j=1}^{n_k^H} m^H_{j,K}(\tau)\,\Psi_j^K \qquad \forall\,\tau \in \mathbf{H}_k^K\,.$$

Now, let $s_H^K : \mathbf{H}_k^K \times \mathbf{H}_k^K \to \mathbb{R}$ be the bilinear form associated with the identity matrix in $\mathbb{R}^{n_k^H \times n_k^H}$, respect to the basis $\{\Psi_j^K\}_{j=1}^{n_k^H}$ of $\mathbf{H}_k^K$. More precisely, we have:

$$s_H^K(\zeta,\tau) \;:=\; \sum_{i=1}^{n_k^H} m^H_{i,K}(\zeta)\, m^H_{i,K}(\tau) \qquad \forall\,\zeta,\,\tau \in \mathbf{H}_k^K\,.$$

Then, we define $\mathcal{S}_H^K : \mathbb{H}_k^K \times \mathbb{H}_k^K \to \mathbb{R}$ as the bilinear form associated with the degrees of freedom of $\mathbb{H}_k^K$ as follows:

$$\mathcal{S}_H^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \sum_{i=1}^{2} s_H^K\big((\zeta_{i1}, \zeta_{i2})^{\mathrm{t}}, (\tau_{i1}, \tau_{i2})^{\mathrm{t}}\big), \tag{2.9}$$

for all $\boldsymbol{\zeta} := (\zeta_{ij})$, $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{H}_k^K$.

In addition, let $P_k^K : L^2(K) \to \mathrm{P}_k(K)$ be the corresponding orthogonal projection, such that, for $v \in L^2(K)$, it is characterized by:

$$P_k^K(v) \in \mathrm{P}_k(K) \text{ and } \int_K P_k^K(v)\, q = \int_K v\, q \qquad \forall\, q \in \mathrm{P}_k(K). \tag{2.10}$$

In turn, let $\mathcal{P}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$ its corresponding vectorial version, such that, for $\mathbf{v} \in \mathbf{L}^2(K)$ there hold:

$$\mathcal{P}_k^K(\mathbf{v}) \in \mathbf{P}_k(K) \text{ and } \int_K \mathcal{P}_k^K(\mathbf{v}) \cdot \mathbf{q} = \int_K \mathbf{v} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_k(K). \tag{2.11}$$

Notice that $\mathcal{P}_k^K(\mathbf{v}) = \big(P_k^K(v_1), P_k^K(v_2)\big)^{\mathrm{t}}$ for all $\mathbf{v} := (v_1, v_2)^{\mathrm{t}} \in \mathbf{L}^2(K)$.

Next, for the following sections, it is important to mention that, according to [5, Section 3.2], the degrees of freedom given in (2.4) allow us the explicit calculation of $\mathcal{P}_k^K(\tau)$ for every $\tau \in \mathbf{H}_k^K$. That is, it is possible to determine the $\mathbf{L}^2(K)$-orthogonal projector for elements in the virtual space. Indeed, it is sufficient to verify that the right-hand side in the second expression of (2.11) can be calculated in these cases. To do that, notice from the definitions of $m_{q,\boldsymbol{n}}^H(\tau)$ and $m_{q,\mathrm{div}}^H(\tau)$ (cf. (2.4)) that it is possible to determine the value of $\mathrm{div}(\tau) \in \mathrm{P}_k(K)$ using the identity:

$$\int_K \mathrm{div}(\tau)\, q = -\int_K \tau \cdot \nabla q + \int_{\partial K} \tau \cdot \boldsymbol{n}\, q \qquad \forall\, q \in \mathrm{P}_k(K). \tag{2.12}$$

Moreover, given $\mathbf{q} \in \mathbf{P}_k(K)$, it is well known that there exist unique $\mathbf{q}^{\perp} \in (\nabla \mathrm{P}_{k+1}(K))^{\perp} \cap \mathbf{P}_k(K)$ and $\widetilde{q} \in \mathrm{P}_{k+1}(K)$, such that $\mathbf{q} = \mathbf{q}^{\perp} + \nabla \widetilde{q}$. In this sense, it follows that:

$$\int_K \tau \cdot \mathbf{q} = \int_K \tau \cdot \mathbf{q}^{\perp} + \int_K \tau \cdot \nabla \widetilde{q} = \int_K \tau \cdot \mathbf{q}^{\perp} - \int_K \widetilde{q}\, \mathrm{div}(\tau) + \int_{\partial K} \tau \cdot \boldsymbol{n}\, \widetilde{q}, \tag{2.13}$$

which, in accordance with (2.12) and the definition of $m_{\mathbf{q},\mathrm{rot}}^H(\tau)$ (see (2.4)), allows the required calculation.

Finally, we also consider the $\mathbb{L}^2(K)$-orthogonal projection $\boldsymbol{\mathcal{P}}_k^K : \mathbb{L}^2(K) \to \mathbb{P}_k(K)$. In other words, $\boldsymbol{\mathcal{P}}_k^K$ is the operator $\mathcal{P}_k^K$ (cf. (2.11)) acting on each row of a tensor of $\mathbb{L}(K)$, which, according to the foregoing discussion, is quite simple to see that $\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})$ can be explicitly calculated for each $\boldsymbol{\tau} \in \mathbb{H}_k^K$.

2.2 $H^1(K)$-conforming subspace, associated bilinear form, and projection

We define the following local virtual element space of order $k \geq 0$ (see, e.g., [21])

$$\begin{aligned} V_k^K \; := \; \Big\{ v \in H^1(K) : \quad & v|_{\partial K} \in E_{k+1}(K)\,, \quad \Delta v \in \mathrm{P}_{k+1}(K)\,, \\ \text{and} \quad & \int_K \big\{ R_k^K(v) - v \big\} q \; = \; 0 \quad \forall\, q \in \widetilde{\mathcal{B}}_k(K) \Big\}\,, \end{aligned} \tag{2.14}$$

where $E_{k+1}(K) := \big\{ v \in C(\partial K) : v|_e \in \mathrm{P}_{k+1}(e) \quad \forall \text{ edge } e \subseteq \partial K \big\}$, $\widetilde{\mathcal{B}}_0(K) := \mathcal{B}_1(K)$, and $\widetilde{\mathcal{B}}_k(K) := \mathcal{B}_{k+1}(K) \setminus \mathcal{B}_{k-1}(K)$ for $k \geq 1$. In addition, $R_k^K : H^1(K) \to \mathrm{P}_{k+1}(K)$ is the projection operator defined for each $v \in H^1(K)$ as the unique polynomial $R_k^K(v) \in \mathrm{P}_{k+1}(K)$ such that (see [22])

$$\begin{aligned} \int_K \nabla R_k^K(v) \cdot \nabla q \; &= \; \int_K \nabla v \cdot \nabla q \qquad \forall\, q \in \mathrm{P}_{k+1}(K)\,, \\ \int_U R_k^K(v) \; &= \; \int_U v\,, \end{aligned} \tag{2.15}$$

where $U = \partial K$ if $k = 0$, and $U = K$ if $k \geq 1$. Now, recalling from [21] the following degrees of freedom for a given $v \in V_k^K$

$$\begin{aligned} m_{i,v}^V(v) \; &:= \; \text{value of } v \text{ at the } i\text{th vertex of } K\,, \quad \forall\, i \text{ vertex of } K\,, \\ m_e^V(v) \; &:= \; \text{values of } v \text{ at } k \text{ u } e\,, \;\; \forall\, e \in \partial K\,, \;\; \text{for } k \geq 1\,, \\ m_{q,K}^V(v) \; &:= \; \text{value of } \int_K vq\,, \;\; \forall\, q \in \mathcal{B}_{k-1}(K)\,, \;\; \text{for } k \geq 1\,, \end{aligned} \tag{2.16}$$

it is easy to check that the dimension of V_k^K is given by

$$\begin{aligned} n_k^V \; = \; n_k^V(K) := d_K \; + \; k\, d_K \; &+ \; \frac{k(k+1)}{2} \\ = \; (k+1) d_K \; &+ \; \frac{k(k+1)}{2}\,. \end{aligned} \tag{2.17}$$

In addition, from [21, Propositions 1 and 2], we know that the set of degrees of freedom from (2.16) is unisolvent in V_k^K.

Next, we now let $\mathbf{V}_k^K$ be the vectorial version of V_k^K given by:

$$\mathbf{V}_k^K \; = \; \Big\{ \mathbf{v} := (v_1, v_2)^{\mathrm{t}} \in \mathbf{H}^1(K) : \quad v_i \in V_k^K \quad \forall\, i \in \{1, 2\} \Big\}\,, \tag{2.18}$$

which, in particular, satisfies that $\dim \mathbf{V}_k^K = 2 n_k^V$. Moreover, let $\mathcal{R}_k^K : \mathbf{H}^1(K) \to \mathbf{P}_{k+1}(K)$ be the vectorial version of the operator R_k^K (cf. (2.15)), that is

$$\begin{aligned} \int_K \nabla \mathcal{R}_k^K(\mathbf{v}) : \nabla \mathbf{q} \; &= \; \int_K \nabla \mathbf{v} : \nabla \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_{k+1}(K)\,, \\ \int_U \mathcal{R}_k^K(\mathbf{v}) \; &= \; \int_U \mathbf{v}\,, \end{aligned}$$

where $U = \partial K$ if $k = 0$, and $U = K$ if $k \geq 1$, which allows to rewrite $\mathbf{V}_k^K$ in the form:

$$\begin{aligned} \mathbf{V}_k^K \;:=\; \Big\{ \mathbf{v} \in \mathbf{H}^1(K) : \quad \mathbf{v}|_{\partial K} \in \mathbf{E}_{k+1}(K)\,, \quad \Delta \mathbf{v} \in \mathbf{P}_{k+1}(K)\,, \\ \text{and} \quad \int_K \big\{ \mathcal{R}_k^K(\mathbf{v}) - \mathbf{v} \big\} \cdot \mathbf{q} \;=\; 0 \quad \forall\, \mathbf{q} \in \widetilde{\boldsymbol{\mathcal{B}}}_k(K) \Big\}\,, \end{aligned}$$

where $\mathbf{E}_{k+1}(K) := [E_{k+1}(K)]^2$, $\widetilde{\boldsymbol{\mathcal{B}}}_0(K) := \boldsymbol{\mathcal{B}}_1(K)$, and $\widetilde{\boldsymbol{\mathcal{B}}}_k(K) := \boldsymbol{\mathcal{B}}_{k+1}(K) \setminus \boldsymbol{\mathcal{B}}_{k-1}(K)$ for $k \geq 1$.

Furthermore, we now denote by $\{m_{j,K}^V(v)\}_{j=1}^{n_k^V}$ the degrees of freedom defined by (2.16), and let $s_V^K : V_k^K \times V_k^K \to \mathbb{R}$ be the associated bilinear form:

$$s_V^K(w, v) \;:=\; \sum_{i=1}^{n_k^V} m_{i,K}^V(w)\, m_{i,K}^V(v) \qquad \forall\, w,\; v \in V_k^K\,,$$

which allows us to define the bilinear form $\mathcal{S}_V^K : \mathbf{V}_k^K \times \mathbf{V}_k^K \to \mathbb{R}$ as follows:

$$\mathcal{S}_V^K(\mathbf{w}, \mathbf{v}) \;:=\; \sum_{i=1}^{2} s_V^K\big(w_i\,, v_i\big)\,, \tag{2.19}$$

for all $\mathbf{w} := (w_1, w_2)^{\mathtt{t}}$, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}} \in \mathbf{V}_k^K$. On the other hand, we introduce $\{\psi_j^K\}_{j=1}^{n_k^V}$ as the canonical basis of V_k^K such that

$$m_{j,K}^V(\psi_i^K) \;=\; \delta_{ij} \qquad \forall\, j = 1, 2, \ldots, n_k^V\,, \tag{2.20}$$

for a given $i = 1, 2, \ldots, n_k^V$. In particular, there holds:

$$v \;=\; \sum_{j=1}^{n_k^V} m_{j,K}^V(v)\, \psi_j^K \qquad \forall\, v \in V_k^K\,.$$

We end this section by clarifying that, for each $\mathbf{v} \in \mathbf{V}_k^K$, its projections $\mathcal{R}_k^K(\mathbf{v})$, $\mathcal{P}_k^K(\mathbf{v})$ and $\boldsymbol{\mathcal{P}}_k^K(\nabla \mathbf{v})$ can be computed explicitly by using the degrees of freedom defined in (2.16). Indeed, using (2.18), it suffices to for each $v \in V_k^K$, to describe how use the degrees of freedom (2.16) to compute $R_k^K(v)$ (cf. (2.15)), $P_k^K(v)$ (cf. (2.10)), and $\mathcal{P}_k^K(\nabla v)$ (cf. (2.11)), respectively. Indeed, we begin by noticing that, for $v \in V_k^K$ and $q \in \mathrm{P}_k(K)$, the right-hand side of the first equation of (2.15) can be integrated by parts to yield

$$\int_K \nabla v \cdot \nabla q \;=\; -\int_K v\, \Delta q \;+\; \int_{\partial K} (\nabla q \cdot \boldsymbol{n})\, v\,, \tag{2.21}$$

where, since $\Delta q \in \mathrm{P}_{k-2}(K)$ and $\nabla q \cdot \boldsymbol{n} \in \mathrm{P}_{k-1}(K)$, the first integral on the right-hand side can be computed by using the degrees of freedom $m_{q,K}^V(v)$, whereas for the second one using $m_{i,v}^V(v)$ and $m_e^V(v)$. Finally, for the right-hand side of the second equation of (2.15), it is straightforward to see that $\int_{\partial K} v = \int_{\partial K} v \cdot 1$ can be calculated using again $m_{i,v}^V(v)$ and $m_e^V(v)$, whereas $\int_K v = \int_K v \cdot 1$ utilizing $m_{q,K}^V(v)$ for $k \geq 1$.

Similarly, integrating by parts we observe that

$$\begin{aligned} \int_K \nabla v \cdot \mathbf{q} &= -\int_K v \operatorname{div}(\mathbf{q}) \, + \int_{\partial K} (\mathbf{q} \cdot \boldsymbol{n}) \, v \\ &= -\int_K P_k^K(v) \operatorname{div}(\mathbf{q}) \, + \int_{\partial K} (\mathbf{q} \cdot \boldsymbol{n}) \, v \qquad \forall \, \mathbf{q} \in \mathbf{P}_k(K) \,, \end{aligned} \tag{2.22}$$

which yields the explicit computation of $\mathcal{P}_k^K(\nabla v)$ for all $v \in V_k^K$.

Finally, for each $v \in V_k^K$, the right-hand side of (2.10) can be computed using the degrees of freedom given by $m_{q,K}^V(v)$ (cf. (2.16)). Indeed, given $q \in \mathrm{P}_k(K)$ we can write $q = \widehat{q} + \widetilde{q}$ such that $\widehat{q} \in \mathcal{B}_{k-1}$ and $\widetilde{q} \in \widetilde{\mathcal{B}}_k$. Thus, $\int_K v \, \widehat{q}$ can be computed using $m_{q,K}^V(v)$, whereas recalling from (2.14) that

$$\int_K v \, \widetilde{q} \; = \; \int_K R_k^K(v) \, \widetilde{q} \,, \tag{2.23}$$

we can compute $\int_K v \, \widetilde{q}$, since $R_k^K(v)$ is explicitly computable for each $v \in V_k^K$.

3 Matrices associated with the projectors

We now aim to describe the explicit calculation of some projections of elements of the virtual subspaces defined in previous section. We begin by remarking that the bases $\{\Psi_j^K\}_{j=1}^{n_k^H}$ and $\{\psi_j^K\}_{j=1}^{n_k^V}$ of $\mathbf{H}_k^K$ (cf. (2.3)) and V_k^K (cf. (2.14)), respectively, are called "virtual" since they are not really known explicitly. Indeed, for example, we do not know precisely if both sets are contained in the polynomial space. We only know some conditions that satisfy their elements on K. More precisely, from the definition of $\mathbf{H}_k^K$, it is quite clear that for every Ψ_j^K, its normal components, divergence, and rotation are known. However, this is not entirely accurate. Indeed, what is really known for each Ψ_j^K are the values of their moments (2.4), which are given in (2.8). In fact, these information are enough to determine, through usual calculations, its normal components, divergence, and rotation, explicitly. We remark in advance that for the following computational implementation, the identities (2.8) (resp. (2.20)) are the only ones required.

In order to perform the implementation of the matrices associated with the projectors, as well as the future matrices associated with the terms arising from virtual schemes, we follow the methodology employed in [20], in which each matrix is assembled through the previous construction of auxiliary matrices intrinsically related to the mixed-VEM method defined by the subspaces $\mathbf{H}_k^K$ and V_k^K (see (2.3) and (2.14), respectively).

3.1 Preliminaries

According to the previous discussion, we need to know polynomial functions that allow us to compute the local operators in a clear way. Thus, consider $K \in \mathcal{T}_h$ and $k \geq 0$. Then, let $\phi_1^e, \phi_2^e, \ldots, \phi_{k+1}^e$ be the basis on the edge $e \in \partial K$ defined in (2.1).

That is, it follows that

$$\phi_i^e(x) \;:=\; \left(\frac{x - x_e}{h_e}\right)^{i-1}, \qquad \text{for } i = 1, 2, \ldots, k+1\,,$$

where x_e is the midpoint of e and h_e its length. In turn, consider $\{\boldsymbol{\phi}_j^e\}_{j=1}^{2(k+1)}$ the vectorial version of the basis $\{\phi_j^e\}_{j=1}^{k+1}$ defined by:

$$\boldsymbol{\phi}_1^e \;:=\; \begin{pmatrix}\phi_1^e\\0\end{pmatrix}, \quad \boldsymbol{\phi}_2^e \;:=\; \begin{pmatrix}\phi_2^e\\0\end{pmatrix}, \quad \ldots, \quad \boldsymbol{\phi}_{k+1}^e \;:=\; \begin{pmatrix}\phi_{k+1}^e\\0\end{pmatrix},$$

$$\boldsymbol{\phi}_{k+2}^e := \begin{pmatrix}0\\\phi_1^e\end{pmatrix}, \quad \boldsymbol{\phi}_{k+3}^e := \begin{pmatrix}0\\\phi_2^e\end{pmatrix}, \quad \ldots, \quad \boldsymbol{\phi}_{2(k+1)}^e := \begin{pmatrix}0\\\phi_{k+1}^e\end{pmatrix}.$$

Now, let $\varphi_1^K, \varphi_2^K, \ldots, \varphi_m^K$, with $m := \frac{(k+1)(k+2)}{2}$, be the basis (2.2) given by:

$$\varphi_i^K(x, y) \;:=\; \left(\frac{x - x_K}{h_K}\right)^{\alpha}\left(\frac{y - y_K}{h_K}\right)^{\beta}, \qquad \text{for } i = 1, 2, \ldots, m\,, \tag{3.1}$$

where $\alpha + \beta \in \{0, 1, \ldots, k\}$, (x_K, y_K) is the barycenter of K and h_K the diameter of K. In particular, we consider

$$i \;:=\; \frac{(\alpha + \beta + 1)(\alpha + \beta + 2)}{2} - \beta\,,$$

where α, β are positive integers, and $\alpha + \beta \leq k$. Using the previous ordering, we can guarantee the hierarchy of the basis $\{\varphi_j^K\}_{j=1}^m$, which indicates that φ_1^K is a constant polynomial. Next, a basis of $\mathbf{P}_k(K)$ is given by:

$$\boldsymbol{\varphi}_1^K \;:=\; \begin{pmatrix}\varphi_1^K\\0\end{pmatrix}, \quad \boldsymbol{\varphi}_2^K \;:=\; \begin{pmatrix}\varphi_2^K\\0\end{pmatrix}, \quad \ldots, \quad \boldsymbol{\varphi}_m^K \;:=\; \begin{pmatrix}\varphi_m^K\\0\end{pmatrix},$$

$$\boldsymbol{\varphi}_{m+1}^K \;:=\; \begin{pmatrix}0\\\varphi_1^K\end{pmatrix}, \quad \boldsymbol{\varphi}_{m+2}^K \;:=\; \begin{pmatrix}0\\\varphi_2^K\end{pmatrix}, \quad \ldots, \quad \boldsymbol{\varphi}_{2m}^K \;:=\; \begin{pmatrix}0\\\varphi_m^K\end{pmatrix},$$

whereas, we consider the following basis of $\mathbb{P}_k(K)$:

$$\Phi_1^K \;:=\; \begin{pmatrix}\varphi_1^K & 0\\0 & 0\end{pmatrix}, \quad \Phi_2^K \;:=\; \begin{pmatrix}\varphi_2^K & 0\\0 & 0\end{pmatrix}, \quad \ldots, \quad \Phi_m^K \;:=\; \begin{pmatrix}\varphi_m^K & 0\\0 & 0\end{pmatrix},$$

$$\Phi_{m+1}^K \;:=\; \begin{pmatrix}0 & \varphi_1^K\\0 & 0\end{pmatrix}, \quad \Phi_{m+2}^K \;:=\; \begin{pmatrix}0 & \varphi_2^K\\0 & 0\end{pmatrix}, \quad \ldots, \quad \Phi_{2m}^K \;:=\; \begin{pmatrix}0 & \varphi_m^K\\0 & 0\end{pmatrix},$$

$$\Phi_{2m+1}^K \;:=\; \begin{pmatrix}0 & 0\\\varphi_1^K & 0\end{pmatrix}, \quad \Phi_{2m+2}^K \;:=\; \begin{pmatrix}0 & 0\\\varphi_2^K & 0\end{pmatrix}, \quad \ldots, \quad \Phi_{3m}^K \;:=\; \begin{pmatrix}0 & 0\\\varphi_m^K & 0\end{pmatrix},$$

$$\Phi_{3m+1}^K \;:=\; \begin{pmatrix}0 & 0\\0 & \varphi_1^K\end{pmatrix}, \quad \Phi_{3m+2}^K \;:=\; \begin{pmatrix}0 & 0\\0 & \varphi_2^K\end{pmatrix}, \quad \ldots, \quad \Phi_{4m}^K \;:=\; \begin{pmatrix}0 & 0\\0 & \varphi_m^K\end{pmatrix}.$$

It is important to clarify that the bases $\{\phi_i^e\}_{i=1}^{k+1}$ and $\{\varphi_i^K\}_{i=1}^m$ are related to the element K. That is, these must be constructed for each element $K \in \mathcal{T}_h$. This is a consequence of the fact that the geometry of every K is not necessarily the same. Then, notice that, from the hierarchical property of the basis $\{\varphi_i^K\}_{i=1}^m$, we can easily extend this basis to degree $k+1$, which will be used throughout this section.

Furthermore, for each $e \in \partial K$, we also require the corresponding Lagrange basis $\mathcal{L}_1^e, \mathcal{L}_2^e, \ldots, \mathcal{L}_{k+2}^e$ on $k+2$ uniformly spaced points of e. More precisely, for each $v \in V_k^K$, using the notation $\alpha_{1,v}^e, \alpha_{2,v}^e, \ldots, \alpha_{k+2,v}^e$ by the $k+2$ uniformly spaced points on e with the corresponding orientation, we have that $m_{a,v}^V(v) = \alpha_{1,v}^e$, $m_e^V(v) \in \{\alpha_{2,v}^e, \alpha_{3,v}^e, \ldots, \alpha_{k+1,v}^e\}$, and $m_{b,v}^V(v) = \alpha_{k+2,v}^e$ (cf. (2.16)), where a and b are the vertices of K that delimit e. Hence, it is well known that there holds

$$v(x) \;=\; \sum_{i=1}^{k+2} \alpha_{i,v}^e \, \mathcal{L}_i^e(x)\,, \qquad \forall\, x \in e\,. \tag{3.2}$$

In turn, we define

$$\boldsymbol{\mathcal{L}}_1^e := \begin{pmatrix} \mathcal{L}_1^e \\ 0 \end{pmatrix}, \quad \boldsymbol{\mathcal{L}}_2^e := \begin{pmatrix} \mathcal{L}_2^e \\ 0 \end{pmatrix}, \quad \ldots, \quad \boldsymbol{\mathcal{L}}_{k+2}^e := \begin{pmatrix} \mathcal{L}_{k+2}^e \\ 0 \end{pmatrix},$$

$$\boldsymbol{\mathcal{L}}_{k+3}^e := \begin{pmatrix} 0 \\ \mathcal{L}_1^e \end{pmatrix}, \quad \boldsymbol{\mathcal{L}}_{k+4}^e := \begin{pmatrix} 0 \\ \mathcal{L}_2^e \end{pmatrix}, \quad \ldots, \quad \boldsymbol{\mathcal{L}}_{2(k+2)}^e := \begin{pmatrix} 0 \\ \mathcal{L}_{k+2}^e \end{pmatrix}.$$

At this point, we remark in advance that we do not need to compute the basis $\{\mathcal{L}_i^e\}_{i=1}^{k+2}$ explicitly, since we only require the Lagrange basis defined on $k+2$ uniformly spaced points of the interval $[0,1]$, which is denoted as:

$$\widehat{\mathcal{L}}_i(t) \;:=\; \prod_{\substack{j=1\\ j\neq i}}^{k+2} \frac{t - t_j}{t_i - t_j}\,, \qquad \forall\, t \in [0,1]\,, \qquad \text{for } i = 1, 2, \ldots, k+2\,, \tag{3.3}$$

where $t_j := \frac{j-1}{k+1}$, with $j = 1, 2, \ldots, k+2$, when positive orientation is considered, and $t_j := 1 - \frac{j-1}{k+1}$, with $j = 1, 2, \ldots, k+2$, otherwise.

On the other hand, we now introduce some matrices in order to facilitate the construction of the discrete operator below. For simplicity, according to the notation introduced at the end of Section 1, if we write $\mathbf{A} := [a_{ij}] \in \mathbb{R}^{p\times q}$, then there holds that $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.

- The mass matrix on an edge e:

  $$\mathbf{M}_{\mathrm{mass},e} \;:=\; \left[\int_e \phi_i^e \phi_j^e\right] \in \mathbb{R}^{(k+1)\times(k+1)},$$

  from which we note that

  $$\int_e \phi_i^e \phi_j^e \;=\; h_e \int_0^1 \widehat{\phi}_i(x)\widehat{\phi}_j(x)\,\mathrm{d}x\,,$$

  where $\widehat{\phi}_i(x) := \left(x - \frac{1}{2}\right)^{i-1}$, for $i = 1, 2, \ldots, k+1$. Thus, the entries in $\mathbf{M}_{\mathrm{mass},e}$ can be determined using a sufficiently precise quadrature rule over $[0,1]$. More precisely, there holds:

  $$\mathbf{M}_{\mathrm{mass},e} \;:=\; h_e \left[\int_0^1 \widehat{\phi}_i(x)\widehat{\phi}_j(x)\,\mathrm{d}x\right] \in \mathbb{R}^{(k+1)\times(k+1)},$$

where the matrix on the right-hand side can be precomputed independently of the edge e, which is important since we also require the matrix:

$$\mathbf{M}^{-1}_{\text{mass},e} := (\mathbf{M}_{\text{mass},e})^{-1} = \frac{1}{h_e}\left[\int_0^1 \widehat{\phi}_i(x)\widehat{\phi}_j(x)\,\mathrm{d}x\right]^{-1}.$$

- The following matrix:

$$\mathbf{M}_{e,K} := \left[\int_e \phi_i^e \varphi_j^K\right] \in \mathbb{R}^{(k+1)\times m_1},$$

with the basis $\{\varphi_i^K\}$ until degree $k+1$, and then $m_1 := \dim \mathrm{P}_{k+1}(K) = \frac{(k+2)(k+3)}{2}$. Next, notice that

$$\int_e \phi_i^e \varphi_j^K = h_e \int_0^1 \widehat{\phi}_i(x)\,\varphi_j^K\big((1-x)\,\mathbf{v}_1 + x\,\mathbf{v}_2\big)\,\mathrm{d}x\,,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vertices of e. Here, the ordering of $\mathbf{v}_1$ and $\mathbf{v}_2$ is according to the orientation of e. For example, we consider $\mathbf{v}_1$ the vertex of e that has the lowest global index in $\mathcal{T}_h$. This selection guarantees to follow a unique orientation on the same edge, independent of the element to which it belongs.
- The mass matrix on an element K:

$$\mathbf{M}^{(k+1)}_{\text{mass},K} := \left[\int_K \varphi_i^K \varphi_j^K\right] \in \mathbb{R}^{m_1\times m_1},$$

which it is calculated for degree $k+1$. In addition, we also require the main submatrix $\mathbf{M}_{\text{mass},K} \in \mathbb{R}^{m\times m}$, whose index range is $[1,m]\times[1,m]$. In other words, $\mathbf{M}^{(k+1)}_{\text{mass},K}$ is the mass matrix for the basis of $\mathrm{P}_{k+1}(K)$, whereas $\mathbf{M}_{\text{mass},K}$ is the mass matrix for the basis of $\mathrm{P}_k(K)$.

Now, in order to compute the entries of $\mathbf{M}^{(k+1)}_{\text{mass},K}$, a quadrature rule over K is not used. Indeed, employing the divergence theorem, we replace the area integral to a sum of line integrals. More precisely, using Gauss's divergence theorem, note that:

$$\begin{aligned}\int_K \left(\frac{x-x_K}{h_K}\right)^{\alpha}\left(\frac{y-y_K}{h_K}\right)^{\beta} &= \frac{h_K}{\alpha+\beta+2}\int_K \operatorname{div}\begin{pmatrix}\left(\frac{x-x_K}{h_K}\right)^{\alpha+1}\left(\frac{y-y_K}{h_K}\right)^{\beta}\\ \left(\frac{x-x_K}{h_K}\right)^{\alpha}\left(\frac{y-y_K}{h_K}\right)^{\beta+1}\end{pmatrix}\\ &= \sum_{e\in\partial K}\frac{h_K}{\alpha+\beta+2}\int_e \left(\frac{x-x_K}{h_K}\right)^{\alpha}\left(\frac{y-y_K}{h_K}\right)^{\beta}\left[\left(\frac{x-x_K}{h_K}\right)n_1^e + \left(\frac{y-y_K}{h_K}\right)n_2^e\right],\end{aligned} \tag{3.4}$$

where $\boldsymbol{n}^e := (n_1^e, n_2^e)^{\mathrm{t}}$ is the unit outward normal at e. Thus, since all entries in $\mathbf{M}^{(k+1)}_{\text{mass},K}$ have the form $\int_K \left(\frac{x-x_K}{h_K}\right)^{\alpha}\left(\frac{y-y_K}{h_K}\right)^{\beta}$, it is enough to use (3.4) to find them (see (3.1)).

- Continuing with the basis $\{\varphi_i^K\}_{i=1}^{m_1}$ of $\mathrm{P}_{k+1}(K)$, the gradient matrix is defined as:

$$\begin{aligned}\mathbf{M}_{\mathrm{grad},K} &:= \left[\int_K \nabla\varphi_{i+1}^K \cdot \nabla\varphi_{j+1}^K\right] \\ &= \left[\int_K \partial_x\varphi_{i+1}^K \partial_x\varphi_{j+1}^K + \partial_y\varphi_{i+1}^K \partial_y\varphi_{j+1}^K\right] \in \mathbb{R}^{(m_1-1)\times(m_1-1)}.\end{aligned} \tag{3.5}$$

  Note that the basis $\{\varphi_i^K\}$ is extended again to degree $k+1$, but eliminating the first constant element. Also, the entries of $\mathbf{M}_{\mathrm{grad},K}$ can be calculated by using the formula (3.4) twice.

Next, for the following matrices, it is important to note that $\{\nabla\varphi_{i+1}^K\}_{i=1}^{m_1-1}$ is a basis of $\nabla\mathrm{P}_{k+1}(K)$. Thus, we now aim to obtain a basis $\{\mathbf{q}_i^K\}_{i=1}^{m_0}$ of $\mathcal{G}_k^\perp(K)$, with $m_0 := \dim \mathrm{P}_{k-1}(K) = \frac{1}{2}k(k+1)$. To do that, it is required to find, for $i = 1, 2, \ldots, m_0$, the constants $\{\alpha_j^{(i)}\}_{j=1}^{2m}$ such that:

$$\mathbf{q}_i^K := \sum_{j=1}^{2m} \alpha_j^{(i)}\, \boldsymbol{\varphi}_j^K \text{ and } \int_K \nabla\varphi_{i+1}^K \cdot \mathbf{q}_j^K = 0\,, \quad \text{for } i = 1, 2, \ldots, m_1 - 1\,.$$

Equivalently, it is required to solve local rectangular linear systems:

$$\mathbf{M}_{0,K}\mathbf{A}_0 = \mathbf{0}_{(m_1-1)\times m_0}\,, \tag{3.6}$$

where $\mathbf{0}_{(m_1-1)\times m_0}$ is the zero matrix of $\mathbb{R}^{(m_1-1)\times m_0}$, and the matrices $\mathbf{M}_{0,K} \in \mathbb{R}^{(m_1-1)\times(2m)}$ and $\mathbf{A}_0 \in \mathbb{R}^{(2m)\times m_0}$ are defined by:

$$\mathbf{M}_{0,K} := \left[\int_K \nabla\varphi_{i+1}^K \cdot \boldsymbol{\varphi}_j^K\right]$$

and

$$\mathbf{A}_0 := \left[\alpha_j^{(\ell)}\right] = \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \cdots & \alpha_1^{(m_0)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \cdots & \alpha_2^{(m_0)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{2m}^{(1)} & \alpha_{2m}^{(2)} & \cdots & \alpha_{2m}^{(m_0)} \end{pmatrix},$$

for all $i = 1, 2, \ldots, m_1 - 1,\ j = 1, 2, \ldots, 2m$ and $\ell = 1, 2, \ldots, m_0$. Regarding $\mathbf{M}_{0,K}$, its construction is similar to $\mathbf{M}_{\mathrm{grad},K}$ (cf. (3.5)). Indeed, it is sufficient to see that $\mathbf{M}_{0,K}$ can be assembled at the following block level:

$$\mathbf{M}_{0,K} = \left[\left[\int_K \partial_x\varphi_{i+1}^K \varphi_j^K\right] \vdots \left[\int_K \partial_y\varphi_{i+1}^K \varphi_j^K\right]\right],$$

where both blocks have size $(m_1 - 1) \times m$. On the other hand, to determine $\mathbf{A}_0$, we need to solve the system (3.6), where *QR* decomposition is a good choice for that.

Alternatively, it is important to mention that the degrees of freedom given by $m_{\mathbf{q},\mathrm{rot}}^H$ (cf. (2.4)) can also be defined employing, instead of $\mathcal{G}_k^\perp(K)$, the basis of any polynomial space $\widetilde{\mathbf{P}}_k(K)$ that satisfies:

$$\mathbf{P}_k(K) = \nabla\mathrm{P}_{k+1}(K) \oplus \widetilde{\mathbf{P}}_k(K)\,.$$

In this way, it is possible to choose a space $\widetilde{\mathbf{P}}_k(K)$ such that the QR decomposition is not required, and then we can calculate these degrees of freedom more efficiently. For more details, we recommend to check [23, Section 2.1], and particularly [23, eq. (2.10)].

At this point, we already introduced some matrices that allow us to construct the operators associated with elements in $\mathbf{H}_k^K$ and $\mathbb{H}_k^K$. However, we will consider discrete schemes that also require the operators associated with elements in V_k^K and $\mathbf{V}_k^K$. According to this, in a similar way as before, we now introduce the following matrices to use below.

- Let $\{e_1, e_2, \ldots, e_{d_K}\}$ be the edges of K. Then, the matrices $\mathbf{C}_L^e\ell \in \mathbb{R}^{(k+2)\times n_k^V}$, with $\ell = 1, 2, \ldots, d_K$, are defined at block level as:

  $$\mathbf{C}_L^e\ell \;:=\; \Big[\, \mathbf{C}_1^{(\ell)} \;\vdots\; \mathbf{0}_{(k+2)\times(k(\ell-1))} \;\vdots\; \mathbf{C}_2 \;\vdots\; \mathbf{0}_{(k+2)\times(k(d_K-\ell)+m_0)} \,\Big],$$

  where n_k^V is defined in (2.17). In addition, matrix $\mathbf{C}_1^{(\ell)} \in \mathbb{R}^{(k+2)\times d_K}$ is identical to the zero matrix $\mathbf{0}_{(k+2)\times d_k}$, except that it has 1 in the entries $(1, \ell)$ and $(k+2, \widehat{d})$, where $\widehat{d} = \ell + 1$ if $\ell \neq d_K$, and $\widehat{d} = 1$ otherwise. Furthermore, matrix $\mathbf{C}_2 \in \mathbb{R}^{(k+2)\times k}$ corresponds to the identity matrix $\mathbf{I}_{k+2}$, whose first and last column were removed.
- For $\ell = 1, 2, \ldots, d_K$, we introduce the following matrices:

  $$\begin{aligned}\mathbf{N}_L^{e_\ell}\ell &:= \left[\int_{e_\ell} \left\{ n_1^{e_\ell}\partial_x\varphi_{i+1}^K + n_2^{e_\ell}\partial_y\varphi_{i+1}^K \right\} \mathcal{L}_j^{e_\ell} \right] \\ &= \left[h_{e_\ell}\int_0^1 \left\{ n_1^{e_\ell}\partial_x\varphi_{i+1}^K(\mathbf{p}_{t,\ell}) + n_2^{e_\ell}\partial_y\varphi_{i+1}^K(\mathbf{p}_{t,\ell}) \right\} \widehat{\mathcal{L}}_j(t)\,\mathrm{d}t \right] \in \mathbb{R}^{(m_1-1)\times(k+2)},\end{aligned}$$

  where $\boldsymbol{n}^{e_\ell} := (n_1^{e_\ell}, n_2^{e_\ell})^{\mathrm{t}}$ is the unit outward normal at e_ℓ, and $\mathbf{p}_{t,\ell} := (1-t)\,\mathbf{v}_1^{(\ell)} + t\,\mathbf{v}_2^{(\ell)}$. Finally, $\mathbf{v}_1^{(\ell)}$ and $\mathbf{v}_2^{(\ell)}$ are the vertices of e_ℓ, sorted according to the orientation of e_ℓ. Moreover, we recall here that $\{\widehat{\mathcal{L}}_j\}_{j=1}^{k+2}$ (cf. (3.3)) depends on the orientation of e_ℓ. Once again, the entries in $\mathbf{N}_L^{e_\ell}\ell$ can be determined using a sufficiently precise quadrature rule over [0, 1].
- The matrix associated with the Laplacian:

  $$\begin{aligned}\mathbf{M}_{\mathrm{lap},K} &:= \left[\int_K \Delta\varphi_{i+1}^K\varphi_j^K\right] \\ &= \left[\int_K \partial_{xx}\varphi_{i+1}^K\varphi_j^K + \partial_{yy}\varphi_{i+1}^K\varphi_j^K\right] \in \mathbb{R}^{(m_1-1)\times m_0}.\end{aligned}$$

  We remark here that the entries of $\mathbf{M}_{\mathrm{lap},K}$ can be calculated by using the formula (3.4) twice.
- The change of basis matrix on e:

  $$\mathbf{M}_{\mathrm{Lag},e} \;:=\; \left[\int_e \mathcal{L}_i^e\,\phi_j^e\right] \;=\; h_e\left[\int_0^1 \widehat{\mathcal{L}}_i(t)\,\widehat{\phi}_j(t)\,\mathrm{d}t\right] \in \mathbb{R}^{(k+2)\times(k+1)},$$

  where, it is important to note that the matrix on the right-hand side can be precomputed independently of the edge e.

- Now, we consider the mass matrix for the Lagrange basis:

  $$\mathbf{M}_{\mathrm{mass},L} := \left[\int_0^1 \widehat{\mathcal{L}}_i(t)\, \widehat{\mathcal{L}}_j(t)\, \mathrm{d}t \right] \in \mathbb{R}^{(k+2)\times(k+2)},$$

  which in fact has two possibilities, one for each possible orientation.

Finally, in order to define the foregoing local matrices, we first recall from the Section 2.1 (see also (2.8)) that $\{\Psi_j^K\}_{j=1}^{n_k^H}$ denotes the canonical basis of $\mathbf{H}_k^K$. Then, let $\{\vec{\boldsymbol{\Psi}}_j^K\}_{j=1}^{2n_k^H}$ be the canonical basis of $\mathbb{H}_k^K$ given by

$$\vec{\boldsymbol{\Psi}}_1^K := \begin{pmatrix} \Psi_1^K \\ \mathbf{0} \end{pmatrix}, \quad \vec{\boldsymbol{\Psi}}_2^K := \begin{pmatrix} \Psi_2^K \\ \mathbf{0} \end{pmatrix}, \quad \ldots, \quad \vec{\boldsymbol{\Psi}}_{n_k^H}^K := \begin{pmatrix} \Psi_{n_k^H}^K \\ \mathbf{0} \end{pmatrix},$$

$$\vec{\boldsymbol{\Psi}}_{n_k^H+1}^K := \begin{pmatrix} \mathbf{0} \\ \Psi_1^K \end{pmatrix}, \quad \vec{\boldsymbol{\Psi}}_{n_k^H+2}^K := \begin{pmatrix} \mathbf{0} \\ \Psi_2^K \end{pmatrix}, \quad \ldots, \quad \vec{\boldsymbol{\Psi}}_{2n_k^H}^K := \begin{pmatrix} \mathbf{0} \\ \Psi_{n_k^H}^K \end{pmatrix}.$$

Similarly, from Section 2.2 (see also (2.20)), we use the basis $\{\psi_j^K\}_{j=1}^{n_k^V}$ of V_k^K to define a basis $\{\vec{\boldsymbol{\psi}}_j^K\}_{j=1}^{2n_k^V}$ of $\mathbf{V}_k^K$ as

$$\vec{\boldsymbol{\psi}}_1^K := \begin{pmatrix} \psi_1^K \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{\psi}}_2^K := \begin{pmatrix} \psi_2^K \\ 0 \end{pmatrix}, \quad \ldots, \quad \vec{\boldsymbol{\psi}}_{n_k^V}^K := \begin{pmatrix} \psi_{n_k^V}^K \\ 0 \end{pmatrix},$$

$$\vec{\boldsymbol{\psi}}_{n_k^V+1}^K := \begin{pmatrix} 0 \\ \psi_1^K \end{pmatrix}, \quad \vec{\boldsymbol{\psi}}_{n_k^V+2}^K := \begin{pmatrix} 0 \\ \psi_2^K \end{pmatrix}, \quad \ldots, \quad \vec{\boldsymbol{\psi}}_{2n_k^V}^K := \begin{pmatrix} 0 \\ \psi_{n_k^V}^K \end{pmatrix}.$$

3.2 The $L^2(K)$-orthogonal projection for elements of $\mathbf{H}_k^K$

Proceeding similarly to [20, Section 5.2], we now aim to describe the implementation of the $\mathbb{L}^2(K)$-orthogonal projection $\boldsymbol{\mathcal{P}}_k^K : \mathbb{L}^2(K) \to \mathbb{P}_k(K)$. More precisely, we introduce the matrix $\mathbf{P}^K \in \mathbb{R}^{(4m)\times(2n_k^H)}$, which performs that projection. In other words, $\mathbf{P}^K$ is a matrix that allows us to approximate the elements of the tensor virtual basis $\{\vec{\boldsymbol{\Psi}}_i^K\}_{i=1}^{2n_k^H}$ through the linear combinations of the elements of the tensor polynomial basis $\{\Phi_i^K\}_{i=1}^{4m}$ by using $\mathbf{P}^K$. It is important to recall that n_k^H was defined in (2.5). More precisely, given $\boldsymbol{\tau} \in \mathbb{H}_k^K$ such that

$$\boldsymbol{\tau} = \sum_{j=1}^{2n_k^H} \alpha_j\, \vec{\boldsymbol{\Psi}}_j^K \text{ and } \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) = \sum_{j=1}^{4m} \beta_j\, \Phi_j^K,$$

we seek a matrix $\mathbf{P}^K \in \mathbb{R}^{(4m)\times(2n_k^H)}$ such that $\boldsymbol{\beta} = \mathbf{P}^K \boldsymbol{\alpha}$. In order to do that, we first consider the following sequential list of matrices:

- $\mathbf{P}_{\mathrm{grad}} := (\mathbf{M}_{0,K})^t\, \mathbf{M}_{\mathrm{grad},K}^{-1} \in \mathbb{R}^{(2m)\times(m_1-1)}$, with $m := \frac{1}{2}(k+1)(k+2)$ and $m_1 := \frac{1}{2}(k+2)(k+3)$.

- $\mathbf{B}_e := s^e\,(\mathbf{M}_{e,K})^{\mathtt{t}}\,\mathbf{M}_{\mathrm{mass},e}^{-1} \in \mathbb{R}^{m_1\times(k+1)}$, for each $e \in \partial K$, where, denoting by I_{v_1} and I_{v_2} the global indexes of the vertexes of e, it follows that:
$$s^e := \begin{cases} 1 & \text{if } I_{v_1} < I_{v_2}, \\ -1 & \text{otherwise,} \end{cases} \tag{3.7}$$
represents an orientation for e, which allows the integrals involved in $\mathbf{M}_{e,K}$ to have the same value independently of the element K. Moreover, the matrix $\mathbf{B}_e^{(r,s)} \in \mathbb{R}^{(s-r+1)\times(k+1)}$ matrix is defined as the submatrix of $\mathbf{B}_e$ that contains all of its columns, but only the rows from r to s, with $1 \le r < s \le m_1$.
- The divergence matrix:
$$\mathbf{M}_{\mathrm{div}} := \mathbf{M}_{\mathrm{mass},K}^{-1}\,\widetilde{\mathbf{B}} \in \mathbb{R}^{m\times n_k^H}, \tag{3.8}$$
with n_k^H defined in (2.5) and $\widetilde{\mathbf{B}} \in \mathbb{R}^{m\times n_k^H}$ is defined at block level as:
$$\widetilde{\mathbf{B}} := \left[\, \mathbf{B}_{e_1}^{(1,m)} \;\vdots\; \mathbf{B}_{e_2}^{(1,m)} \;\vdots\; \cdots \;\vdots\; \mathbf{B}_{e_{d_k}}^{(1,m)} \;\vdots\; -\widetilde{\mathbf{C}} \;\vdots\; \mathbf{0}_{m\times m_0} \,\right],$$
where $m_0 := \frac{1}{2}k(k+1)$. Furthermore, the matrix $\widetilde{\mathbf{C}} \in \mathbb{R}^{m\times(m-1)}$ is identical to the identity matrix $\mathbf{I}_m$, except that its first column was removed.
- The matrix $\mathrm{Pr} \in \mathbb{R}^{(2m)\times n_k^H}$, which is associated to the right-hand side of (2.11). More precisely, using (2.13) we define Pr as follows:
$$\begin{aligned} \mathrm{Pr} &:= \left[\int_K \boldsymbol{\varphi}_i^K \cdot \Psi_j^K\right] = \left[-\int_K \widetilde{\varphi_i^K}\,\mathrm{div}(\Psi_j^K)\right] + \left[\int_{\partial K} \widetilde{\varphi_i^K}\,(\Psi_j^K \cdot \boldsymbol{n})\right] + \left[\int_K (\boldsymbol{\varphi}_i^K)^{\perp} \cdot \Psi_j^K\right] \\ &= -\mathbf{P}_{\mathrm{grad}}\,\widetilde{\mathbf{M}}_{\mathrm{mass}}\,\mathbf{M}_{\mathrm{div}} + \mathbf{P}_{\mathrm{grad}}\left[\, \mathbf{B}_{e_1}^{(2,m1)} \;\vdots\; \mathbf{B}_{e_2}^{(2,m1)} \;\vdots\; \cdots \;\vdots\; \mathbf{B}_{e_{d_k}}^{(2,m1)} \;\vdots\; \mathbf{0}_{(m1-1)\times(m-1+m_0)} \,\right] \\ &\quad + \left[\, \mathbf{0}_{(2m)\times((k+1)d_K+m-1)} \;\vdots\; (\mathbf{I}_{\mathrm{Mass}} - \mathbf{P}_{\mathrm{grad}}\,\mathbf{M}_{0,K})\,\mathbf{A}_0\,(\mathbf{A}_0^{\mathtt{t}}\,\mathbf{I}_{\mathrm{Mass}}\,\mathbf{A}_0)^{-1} \,\right], \end{aligned}$$
where $\widetilde{\mathbf{M}}_{\mathrm{mass}} \in \mathbb{R}^{(m_1-1)\times m}$ is the submatrix of $\mathbf{M}_{\mathrm{mass},K}^{(k+1)}$, which considers the rows from 2 to m_1 and columns from 1 to m. In turn, it follows that:
$$\mathbf{I}_{\mathrm{Mass}} := \mathrm{kron}(\mathbf{I}_2,\, \mathbf{M}_{\mathrm{mass},K}) \in \mathbb{R}^{(2m)\times(2m)},$$
where $\mathrm{kron}(\cdot,\cdot)$ corresponds to the usual Kronecker product.

Finally, using the foregoing matrices, we assemble the following matrix:
$$\mathbf{P}^K t := \mathrm{kron}(\mathbf{I}_2,\, \mathbf{M}_{\mathrm{mass},K}^{-1})\,\mathrm{Pr} \in \mathbb{R}^{(2m)\times n_k^H}, \tag{3.9}$$
which corresponds to the coefficients of the elements of $\{\Psi_i^K\}_{i=1}^{n_k^H}$ under the operator $\mathcal{P}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$. On the other hand, the matrix of coefficients of the elements of $\{\vec{\boldsymbol{\Psi}}_i^K\}_{i=1}^{2n_k^H}$ under the operator $\boldsymbol{\mathcal{P}}_k^K : \mathbb{L}(K) \to \mathbb{P}_k(K)$ is given by:
$$\mathbf{P}^K := \mathrm{kron}(\mathbf{I}_2,\, \hat{\mathrm{P}}_0) \in \mathbb{R}^{(4m)\times(2n_k^H)}.$$

3.3 The projection operator R_k^K for elements of V_k^K

Concerning the projection $R_k^K : H^1(K) \to \mathrm{P}_{k+1}(K)$ (cf. (2.15)), we now aim to define the matrix $\mathbf{R}^K$ associated to the coefficients of the elements of $\{\psi_i^K\}_{i=1}^{n_k^V}$ under the operator R_k^K. Firstly, we define the matrices $\mathbf{B}_1^R, \mathbf{B}_2^R \in \mathbb{R}^{(m_1-1)\times n_k^V}$ as follows:

$$\mathbf{B}_1^R := \left[-\int_K \Delta\varphi_{i+1}^K \, \psi_j^K \right] = -\mathbf{M}_{\mathrm{lap},K} \left(\mathbf{M}_{\mathrm{mass},K}^{(k-1)}\right)^{-1} \left[\mathbf{0}_{m_0\times((k+1)d_K)} \,\vdots\, \mathbf{I}_{m_0} \right],$$

and

$$\mathbf{B}_2^R := \left[\int_{\partial K} (\nabla\varphi_{i+1}^K \cdot \boldsymbol{n}) \, \psi_j^K \right] = \sum_{\ell=1}^{d_K} \mathbf{N}_L^{e_\ell}\,\ell\,\mathbf{C}_L^{e}\,\ell,$$

where we use (3.2) with $v = \psi_j^K$. In addition, $\mathbf{M}_{\mathrm{mass},K}^{(k-1)} \in \mathbb{R}^{m_0\times m_0}$ is the submatrix of $\mathbf{M}_{\mathrm{mass},K}^{(k+1)}$, which considers the rows and columns from 1 to m_0.

Now, from (2.21), we easily realize that the first equation in (2.15) is associated to the matrix

$$\widehat{\mathbf{R}}_0 := \mathbf{M}_{\mathrm{grad},K}^{-1} (\mathbf{B}_1^R + \mathbf{B}_2^R) \in \mathbb{R}^{(m_1-1)\times n_k^V}, \tag{3.10}$$

whereas the second one is implemented as:

$$\eta_1 := \begin{cases} \left(\displaystyle\sum_{e\in\partial K} h_e\right)^{-1} \displaystyle\sum_{e\in\partial K} \left\{ h_e\, \mathbf{c}_0\, \mathbf{C}_L^e - \mathbf{c}_1 \widehat{\mathbf{R}}_0 \right\} & \text{if } k = 0, \\ \dfrac{1}{|K|} \left\{ \left[\mathbf{0}_{1\times((k+1)d_K)} \,\vdots\, 1 \,\vdots\, \mathbf{0}_{1\times(m_0-1)} \right] - \mathbf{c}_2\, \widehat{\mathbf{R}}_0 \right\} & \text{otherwise,} \end{cases}$$

where $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^{1\times m_1-1}$ are the first row of $\mathbf{M}_{e,K} \in \mathbb{R}^{(k+1)\times m_1}$ and $\mathbf{M}_{\mathrm{mass},K}^{(k+1)} \in \mathbb{R}^{m_1\times m_1}$, respectively, where their first entry has been removed. Moreover, the vector $\mathbf{c}_0 := \left[\int_0^1 \widehat{\mathcal{L}}_j(t)\,\mathrm{d}t \right] \in \mathbb{R}^{1\times(k+2)}$ can be precomputed depending only of the orientation of e.

We end this section by introducing the matrix of coefficients of the elements of $\{\psi_i^K\}_{i=1}^{n_k^V}$ under the operator $R_k^K : H^1(K) \to \mathrm{P}_{k+1}(K)$ which is given by:

$$\mathbf{R}^K := \left[\begin{array}{c} \eta_1 \\ \hline \widehat{\mathbf{R}}_0 \end{array} \right] \in \mathbb{R}^{m_1\times n_k^V}.$$

It is important to remark here that the matrix of coefficients of the elements of $\{\vec{\boldsymbol{\psi}}_i^K\}_{i=1}^{2n_k^V}$ under the operator $\mathcal{R}_k^K : \mathbf{H}^1(K) \to \mathbf{P}_{k+1}(K)$ can be obtained by $\mathrm{kron}(\mathbf{I}_2,\, \mathbf{R}^K) \in \mathbb{R}^{(2m_1)\times(2n_k^V)}$.

3.4 The $L^2(K)$-orthogonal projection for elements of V_k^K

Similarly to the previous section, we now aim to introduce the matrices associated with the projection $P_k^K : L^2(K) \to \mathrm{P}_k(K)$ for elements of the canonical basis of

V_k^K, and the projection $\boldsymbol{\mathcal{P}}_k^K : \mathbb{L}(K) \to \mathbb{P}_k(K)$ for the gradient of elements of the canonical basis of $\mathbf{V}_k^K$. Indeed, we begin by considering the matrix of coefficients of the elements of $\{\psi_i^K\}_{i=1}^{n_k^V}$ under the operator P_k^K, which employing (2.23) is given by:

$$\begin{aligned} \mathbf{P}_V \; &:= \; \left[\int_K \varphi_i^K \, \varphi_j^K\right]^{-1} \left[\int_K \varphi_i^K \, \psi_j^K\right] \\ &= \; \mathbf{M}_{\mathrm{mass},K}^{-1} \left[\begin{array}{c} \left[\, \mathbf{0}_{m_0 \times ((k+1) d_K)} \;\vdots\; \mathbf{I}_{m_0} \,\right] \\ \hdashline \widetilde{\mathbf{U}}\, \mathbf{R}^K \end{array}\right] \in \mathbb{R}^{m \times n_k^V} \, , \end{aligned} \tag{3.11}$$

where the matrix $\widetilde{\mathbf{U}} \in \mathbb{R}^{(m-m_0)\times m_1}$ is the submatrix of $\mathbf{M}_{\mathrm{mass},K}^{(k+1)}$, whose index range is $[m_0+1, m] \times [1, m_1]$. Now, we follow (2.22) and let $\mathcal{D}\mathbf{P}_V \in \mathbb{R}^{(4m)\times(2n_k^V)}$ be the matrix of coefficients of the elements of $\{\nabla \vec{\boldsymbol{\psi}}_i^K\}_{i=1}^{2n_k^V}$ under the operator $\boldsymbol{\mathcal{P}}_k^K$. Hence, we deduce that:

$$\begin{aligned} \mathcal{D}\mathbf{P}_V &:= \left[\int_K \Phi_i^K : \Phi_j^K\right]^{-1} \left[\int_K \Phi_i^K : \nabla \vec{\boldsymbol{\psi}}_j^K\right] \\ &= \; \mathrm{kron}\left(\mathbf{I}_4, \, \mathbf{M}_{\mathrm{mass},K}^{-1}\right) \mathrm{kron}\left(\mathbf{I}_2, \, \left[\int_K \boldsymbol{\varphi}_i^K \cdot \nabla \psi_j^K\right]\right) \\ &= \; \mathrm{kron}\left(\mathbf{I}_4, \, \mathbf{M}_{\mathrm{mass},K}^{-1}\right) \mathrm{kron}\left(\mathbf{I}_2, \, -\left[\int_K \mathrm{div}(\boldsymbol{\varphi}_i^K)\, P_k^K(\psi_j^K)\right] + \left[\int_{\partial K} (\boldsymbol{\varphi}_i^K \cdot \boldsymbol{n})\, \psi_j^K\right]\right) \\ &= \; \mathrm{kron}\left(\mathbf{I}_4, \, \mathbf{M}_{\mathrm{mass},K}^{-1}\right) \mathrm{kron}\left(\mathbf{I}_2, \, -\left[\int_K \mathrm{div}(\boldsymbol{\varphi}_i^K)\, \varphi_j^K\right] \mathbf{P}_V + \sum_{e \in \partial K} \mathrm{kron}\left(\boldsymbol{n}^e, \, \left[\int_e \varphi_i^K \, \mathcal{L}_j^e\right] \mathbf{C}_L^e\right)\right), \end{aligned}$$

where $\boldsymbol{n}^e := (n_1^e, n_2^e)^{\mathrm{t}}$ is the unit outward normal at e. In addition, we introduce the following matrices:

$$\mathbf{P}_{\mathrm{div}} \; := \; \left[\int_K \mathrm{div}(\boldsymbol{\varphi}_i^K)\, \varphi_j^K\right] \; = \; \left[\begin{array}{c} \mathbf{0}_{1\times m} \\ \hdashline \mathbf{D}_x \\ \hdashline \mathbf{0}_{1\times m} \\ \hdashline \mathbf{D}_y \end{array}\right] \in \mathbb{R}^{(2m)\times m} \, ,$$

and

$$\mathbf{P}_L^e \; := \; \left[\int_e \varphi_i^K \, \mathcal{L}_j^e\right] \; = \; \left[h_e \int_0^1 \varphi_i^K((1-t)\,\mathbf{v}_1 + t\,\mathbf{v}_2)\, \widehat{\mathcal{L}}_j(t)\, \mathrm{d}t \right] \in \mathbb{R}^{m \times (k+2)} \, ,$$

where $\mathbf{D}_x \in \mathbb{R}^{(m-1)\times m}$ is the submatrix of $\mathbf{M}_{0,K}$ with index range $[1, m-1] \times [1, m]$, whereas $\mathbf{D}_y \in \mathbb{R}^{(m-1)\times m}$ is the submatrix of $\mathbf{M}_{0,K}$ with index range $[1, m-1] \times [m+1, 2m]$. Moreover, once again, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the oriented vertices of e. Then, according to the previous analysis, we conclude that

$$\mathcal{D}\mathbf{P}_V \; = \; \mathrm{kron}\left(\mathbf{I}_4, \, \mathbf{M}_{\mathrm{mass},K}^{-1}\right) \mathrm{kron}\left(\mathbf{I}_2, \, -\mathbf{P}_{\mathrm{div}}\, \mathbf{P}_V \, + \, \sum_{e \in \partial K} \mathrm{kron}\left(\boldsymbol{n}^e, \, \mathbf{P}_L^e \, \mathbf{C}_L^e\right)\right) \in \mathbb{R}^{(4m)\times(2n_k^V)} \, .$$

4 Local discrete operators arising from VEM schemes

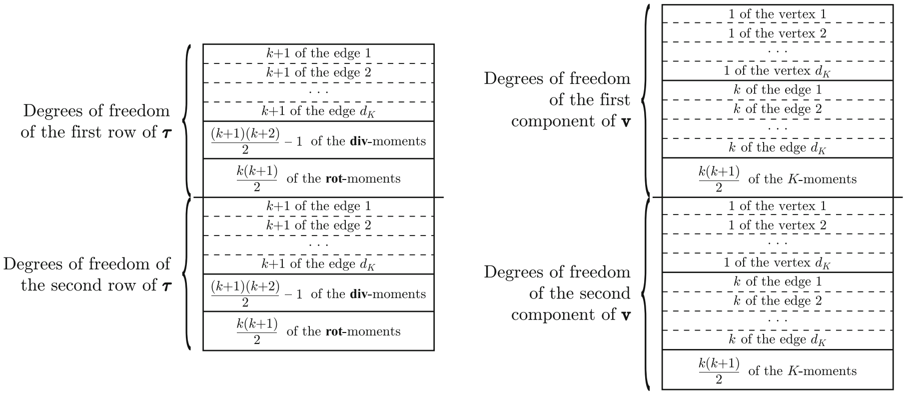
We begin by describing in detail a way to assemble diverse local terms arising in mixed-primal virtual element formulations as in [7–9, 24–26]. In particular, we are interested in those coming from the scheme proposed in [8] for the Navier-Stokes system, which is recalling in Section 5.2 below. For simplicity, we define three categories to separate the terms (or operators) related to the two virtual subspaces $\mathbb{H}_k^K$ and $\mathbf{V}_k^K$, as well as those that combine both. Therein, matrices and vectors are described locally to eventually be assembled to the respective global linear system. A fourth category related to particular nonlinear schemes (such as Navier-Stokes or Boussinesq) is presented later in the Section 5.4.

On the other hand, for each $K \in \mathcal{T}_h$, in what follows we sort the degrees of freedom of each element $\boldsymbol{\tau} \in \mathbb{H}_k^K$ as in Fig. 1a, whereas the moments of each element $\mathbf{v} \in \mathbf{V}_k^K$ follow the ordering described in Fig. 1b.

4.1 Operators related to the elements of $\mathbb{H}_k^K$

Given a volume force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and a Dirichlet datum $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we consider the following discrete operators:

- $\mathbf{A}_{\text{dev}}^K := \left[\displaystyle\int_K \left[\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K)\right]^{\mathrm{d}} : \left[\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\right]^{\mathrm{d}}\right] \in \mathbb{R}^{2n_k^H \times 2n_k^H}$
- $\mathbf{A}_{\text{tra}}^K := \left[\displaystyle\int_K \operatorname{tr}\big(\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K)\big)\operatorname{tr}\big(\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\big)\right] \in \mathbb{R}^{2n_k^H \times 2n_k^H}$
- $\mathbf{A}_{\text{div}}^K := \left[\displaystyle\int_K \mathbf{div}(\vec{\boldsymbol{\Psi}}_i^K) \cdot \mathbf{div}(\vec{\boldsymbol{\Psi}}_j^K)\right] \in \mathbb{R}^{2n_k^H \times 2n_k^H}$
- $\mathbf{A}_{\text{sta}}^K := \left[\mathcal{S}_H^K\big(\vec{\boldsymbol{\Psi}}_i^K - \boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K), \vec{\boldsymbol{\Psi}}_j^K - \boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\big)\right] \in \mathbb{R}^{2n_k^H \times 2n_k^H}$



(a) (b)

Fig. 1 Ordering of local degrees of freedom for each $K \in \mathcal{T}_h$

- $\mathbf{a}_{\mathrm{tra}}^K := \left[\int_K \mathrm{tr}(\vec{\boldsymbol{\Psi}}_i^K)\right] \in \mathbb{R}^{2n_k^H \times 1}$
- $\mathbf{b}_1^K := \left[\int_{\partial K \cap \Gamma} \vec{\boldsymbol{\Psi}}_i^K \boldsymbol{n} \cdot \mathbf{g}\right] \in \mathbb{R}^{2n_k^H \times 1}$
- $\mathbf{b}_2^K := \left[\int_K \mathbf{div}(\vec{\boldsymbol{\Psi}}_i^K) \cdot \mathbf{f}\right] \in \mathbb{R}^{2n_k^H \times 1}$

where the bilinear form $\mathcal{S}_H^K(\cdot,\cdot)$ is defined in (2.9). Now, the operator $\mathbf{A}_{\mathrm{dev}}^K \in \mathbb{R}^{2n_k^H \times 2n_k^H}$ is defined by:

$$\begin{aligned}\mathbf{A}_{\mathrm{dev}}^K &:= \left[\int_K \left[\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K)\right]^{\mathtt{d}} : \left[\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\right]^{\mathtt{d}}\right] = (\mathbf{P}^K)^{\mathtt{t}} \left[\int_K (\Phi_i^K)^{\mathtt{d}} : (\Phi_j^K)^{\mathtt{d}}\right] \mathbf{P}^K \\ &= (\mathbf{P}^K)^{\mathtt{t}} \,\mathrm{kron}(\mathbf{M}_{\mathrm{dev}},\, \mathbf{M}_{\mathrm{mass},K})\, \mathbf{P}^K,\end{aligned}$$

where

$$\mathbf{M}_{\mathrm{dev}} := \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \tag{4.1}$$

is the matrix associated to the deviator operator. Similarly, for the operator $\mathbf{A}_{\mathrm{tra}}^K \in \mathbb{R}^{2n_k^H \times 2n_k^H}$, it follows:

$$\begin{aligned}\mathbf{A}_{\mathrm{tra}}^K &:= \left[\int_K \mathrm{tr}\big(\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K)\big)\, \mathrm{tr}\big(\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\big)\right] = (\mathbf{P}^K)^{\mathtt{t}} \left[\int_K \mathrm{tr}(\Phi_i^K)\, \mathrm{tr}(\Phi_j^K)\right] \mathbf{P}^K \\ &= (\mathbf{P}^K)^{\mathtt{t}} \,\mathrm{kron}\left(\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},\, \mathbf{M}_{\mathrm{mass},K}\right) \mathbf{P}^K,\end{aligned}$$

whereas, the operator $\mathbf{A}_{\mathrm{div}}^K \in \mathbb{R}^{2n_k^H \times 2n_k^H}$ can be computed as:

$$\begin{aligned}\mathbf{A}_{\mathrm{div}}^K &:= \left[\int_K \mathbf{div}(\vec{\boldsymbol{\Psi}}_i^K) \cdot \mathbf{div}(\vec{\boldsymbol{\Psi}}_j^K)\right] = \mathrm{kron}\left(\mathbf{I}_2,\, \left[\int_K \mathrm{div}(\Psi_i^K)\, \mathrm{div}(\Psi_j^K)\right]\right) \\ &= \mathrm{kron}\left(\mathbf{I}_2,\, (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}} \left[\int_K \varphi_i^K \varphi_j^K\right] \mathbf{M}_{\mathrm{div}}\right) = \mathrm{kron}\left(\mathbf{I}_2,\, (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}}\, \mathbf{M}_{\mathrm{mass},K}\, \mathbf{M}_{\mathrm{div}}\right).\end{aligned}$$

On the other hand, the operator $\mathbf{A}_{\text{sta}}^K \in \mathbb{R}^{2n_k^H \times 2n_k^H}$ is assembled in a different way. More precisely, from (2.9), we have that

$$\mathbf{A}_{\text{sta}}^K := \left[\mathcal{S}_H^K\big(\vec{\boldsymbol{\Psi}}_i^K - \mathcal{P}_k^K(\vec{\boldsymbol{\Psi}}_i^K), \vec{\boldsymbol{\Psi}}_j^K - \mathcal{P}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\big)\right] = \text{kron}(\mathbf{I}_2, \mathbf{H}_A^{\text{t}}\, \mathbf{H}_A),$$

where $\mathbf{H}_A \in \mathbb{R}^{n_k^H \times n_k^H}$ can be decomposed as follows: $\mathbf{H}_A = \mathbf{H}_1 - \mathbf{H}_2\, \mathbf{P}^K t$. Here, $\mathbf{P}^K t \in \mathbb{R}^{(2m)\times n_k^H}$ is defined in (3.9), whereas $\mathbf{H}_1 \in \mathbb{R}^{n_k^H \times n_k^H}$ is a diagonal matrix given at block level by:

$$\mathbf{H}_1 := \left[\begin{array}{c:c:c:c:c} s^{e_1}\,\mathbf{I}_{k+1} & & & & \\ \hdashline & s^{e_2}\,\mathbf{I}_{k+1} & & & \\ \hdashline & & \ddots & & \\ \hdashline & & & s^{e_{d_K}}\,\mathbf{I}_{k+1} & \\ \hdashline & & & & \mathbf{0}_{m-1+m_0} \end{array}\right],$$

where the constant s^e is defined in (3.7), and $m_0 := \frac{1}{2}k(k+1)$. Furthermore, for each $e \in \partial K$, let $\mathbf{H}_2^e e \in \mathbb{R}^{(k+1)\times(2m)}$ be the matrix given by:

$$\mathbf{H}_2^e e := \text{kron}(\boldsymbol{n}^e, \widetilde{\mathbf{M}}_{e,K}),$$

where $\boldsymbol{n}^e \in \mathbb{R}^{1\times 2}$ is the unit outward row normal at e and $\widetilde{\mathbf{M}}_{e,K} \in \mathbb{R}^{(k+1)\times m}$ is the submatrix of $\mathbf{M}_{e,K}$, which contains all its rows, but only the columns from 1 to m. In this way, it follows that $\mathbf{H}_2 \in \mathbb{R}^{n_k^H \times (2m)}$ is defined as:

$$\mathbf{H}_2 := \left[\begin{array}{c} \mathbf{H}_2^e e_1 \\ \hdashline \mathbf{H}_2^e e_2 \\ \hdashline \vdots \\ \hdashline \mathbf{H}_2^e e_{d_K} \\ \hdashline \mathbf{0}_{(m-1+m_0)\times(2m)} \end{array}\right].$$

Furthermore, the operator $\mathbf{a}_{\text{tra}}^K \in \mathbb{R}^{2n_k^H \times 1}$ satisfies that:

$$\mathbf{a}_{\text{tra}}^K := \left[\int_K \text{tr}(\vec{\boldsymbol{\Psi}}_i^K)\right] = (\mathbf{P}^K)^{\text{t}} \left[\int_K \text{tr}(\Phi_i^K)\right] = (\mathbf{P}^K)^{\text{t}}\, \text{kron}\left(\begin{matrix}1\\0\\0\\1\end{matrix}, \mathbf{c}_3\right),$$

where the vector $\mathbf{c}_3 \in \mathbb{R}^{m\times 1}$ is the first column of $\mathbf{M}_{\text{mass},K}$.

Finally, letting $\mathbf{g} := (g_1, g_2)^{\mathrm{t}}$ be the Dirichlet datum, and given a boundary edge e, we define the following vectors for the coefficients of the $L^2(e)$-orthogonal projection:

$$\begin{aligned} \mathbf{p}^{e}_{g_{\ell}}\ell &:= \left[\int_e \phi^e_i\, \phi^e_j\right]^{-1} \left[\int_e \phi^e_i\, g_\ell\right] \\ &= \mathbf{M}^{-1}_{\mathrm{mass},e} \left[h_e \int_0^1 \widehat{\phi}_i(x)\, g_\ell((1-x)\,\mathbf{v}_1 + x\,\mathbf{v}_2)\,\mathrm{d}x\right] \in \mathbb{R}^{(k+1)\times 1}\,, \end{aligned} \tag{4.2}$$

for $\ell = 1, 2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the oriented vertices of e. Next, note that the operator $\mathbf{b}^K_1 \in \mathbb{R}^{2n^H_k\times 1}$ is given by:

$$\mathbf{b}^K_1 := \left[\int_{\partial K\cap\Gamma} \vec{\boldsymbol{\Psi}}^K_i\, \boldsymbol{n}\cdot\mathbf{g}\right] = \sum_{e\in\partial K\cap\Gamma} \left[\begin{array}{c} \left[\displaystyle\int_e (\Psi^K_i\cdot\boldsymbol{n}^e)\, g_1\right] \\ \hdashline \left[\displaystyle\int_e (\Psi^K_i\cdot\boldsymbol{n}^e)\, g_2\right] \end{array}\right]$$

$$= \sum_{e\in\partial K\cap\Gamma} \left[\begin{array}{c} \left[\displaystyle\int_e (\Psi^K_i\cdot\boldsymbol{n}^e)\, \phi^e_j\right] \mathbf{p}^{e}_{g_{\ell}}1 \\ \hdashline \left[\displaystyle\int_e (\Psi^K_i\cdot\boldsymbol{n}^e)\, \phi^e_j\right] \mathbf{p}^{e}_{g_{\ell}}2 \end{array}\right] = \left[\begin{array}{c} \mathbf{b}^{e_1}_{1,1} \\ \hdashline \vdots \\ \hdashline \mathbf{b}^{e_{d_K}}_{1,1} \\ \hdashline \mathbf{0}_{(m-1+m_0)\times 1} \\ \hdashline\hdashline \mathbf{b}^{e_1}_{1,2} \\ \hdashline \vdots \\ \hdashline \mathbf{b}^{e_{d_K}}_{1,2} \\ \hdashline \mathbf{0}_{(m-1+m_0)\times 1} \end{array}\right],$$

where $\mathbf{b}^e_{1,\ell} \in \mathbb{R}^{(k+1)\times 1}$, for each $e \in \partial K$, is defined as:

$$\mathbf{b}^e_{1,\ell} := \begin{cases} s^e\, \mathbf{p}^{e}_{g_{\ell}}\ell & \text{if } e \in \Gamma, \\ \mathbf{0}_{(k+1)\times 1} & \text{otherwise,} \end{cases}$$

for $\ell = 1, 2$, where s^e is defined in (3.7). Now, in a similar way, we consider $\mathbf{f} := (f_1, f_2)^{\mathrm{t}}$ the source term, and then introduce the following vectors for the coefficients of the $L^2(K)$-orthogonal projection:

$$\mathbf{p}^{K}_{f_{\ell}}\ell := \left[\int_K \varphi^K_i\, \varphi^K_j\right]^{-1} \left[\int_K \varphi^K_i\, f_\ell\right] = \mathbf{M}^{-1}_{\mathrm{mass},K} \left[\int_K \varphi^K_i\, f_\ell\right] \in \mathbb{R}^{m\times 1}\,, \tag{4.3}$$

where the integral $\int_K \varphi_i^K\, f_\ell$ requires a suitable quadrature for polygonal domains. Some examples can be found in [27–29]. Hence, it follows that

$$\mathbf{b}_2^K := \left[\int_K \mathbf{div}(\vec{\mathbf{\Psi}}_i^K)\cdot \mathbf{f}\right] = \left[\begin{array}{c} \left[\int_K \mathrm{div}(\Psi_i^K)\, P_k^K(f_1)\right] \\ \hdashline \left[\int_K \mathrm{div}(\Psi_i^K)\, P_k^K(f_2)\right] \end{array}\right]$$

$$= \left[\begin{array}{c} (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}} \left[\int_K \varphi_i^K\, \varphi_j^K\right] \mathbf{p}_{f_\ell}^K 1 \\ \hdashline (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}} \left[\int_K \varphi_i^K\, \varphi_j^K\right] \mathbf{p}_{f_\ell}^K 2 \end{array}\right] = \left[\begin{array}{c} (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}}\, \mathbf{M}_{\mathrm{mass},K}\, \mathbf{p}_{f_\ell}^K 1 \\ \hdashline (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}}\, \mathbf{M}_{\mathrm{mass},K}\, \mathbf{p}_{f_\ell}^K 2 \end{array}\right]$$

$$= \left[\begin{array}{c} (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}} \left[\int_K \varphi_i^K\, f_1\right] \\ \hdashline (\mathbf{M}_{\mathrm{div}})^{\mathtt{t}} \left[\int_K \varphi_i^K\, f_2\right] \end{array}\right] \in \mathbb{R}^{(2n_k^H)\times 1},$$

where $\mathbf{M}_{\mathrm{div}} \in \mathbb{R}^{m\times n_k^H}$ is defined in (3.8).

4.2 Operators related to the elements of $\mathbf{V}_k^K$

Considering again $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, together with the bilinear form $\mathcal{S}_V^K(\cdot,\cdot)$ (see (2.19)), we now consider the operators:

- $\mathbf{D}_{\mathrm{gra}}^K := \left[\int_K \nabla\mathcal{R}_k^K(\vec{\boldsymbol{\psi}}_i^K) : \nabla\mathcal{R}_k^K(\vec{\boldsymbol{\psi}}_j^K)\right] \in \mathbb{R}^{2n_k^V\times 2n_k^V}$
- $\mathbf{D}_{\mathrm{sta}}^K := \left[\mathcal{S}_V^K\left(\vec{\boldsymbol{\psi}}_i^K - \mathcal{R}_k^K(\vec{\boldsymbol{\psi}}_i^K),\, \vec{\boldsymbol{\psi}}_j^K - \mathcal{R}_k^K(\vec{\boldsymbol{\psi}}_j^K)\right)\right] \in \mathbb{R}^{2n_k^V\times 2n_k^V}$
- $\mathbf{D}_{\mathrm{bou}}^K := \left[\int_{\partial K\cap\Gamma} \vec{\boldsymbol{\psi}}_i^K \cdot \vec{\boldsymbol{\psi}}_j^K\right] \in \mathbb{R}^{2n_k^V\times 2n_k^V}$
- $\mathbf{b}_3^K := \left[\int_{\partial K\cap\Gamma} \vec{\boldsymbol{\psi}}_i^K \cdot \mathbf{g}\right] \in \mathbb{R}^{2n_k^V\times 1}$
- $\mathbf{b}_4^K := \left[\int_K \vec{\boldsymbol{\psi}}_i^K \cdot \mathcal{P}_k^K(\mathbf{f})\right] \in \mathbb{R}^{2n_k^V\times 1}$

In turn, we begin by describing the operator $\mathbf{D}_{\mathrm{gra}}^K \in \mathbb{R}^{2n_k^V\times 2n_k^V}$ as follows:

$$\mathbf{D}_{\mathrm{gra}}^K := \left[\int_K \nabla\mathcal{R}_k^K(\vec{\boldsymbol{\psi}}_i^K) : \nabla\mathcal{R}_k^K(\vec{\boldsymbol{\psi}}_j^K)\right] = \mathrm{kron}\left(\mathbf{I}_2,\, \left[\int_K \nabla R_k^K(\psi_i^K)\cdot\nabla R_k^K(\psi_j^K)\right]\right)$$

$$= \mathrm{kron}\left(\mathbf{I}_2,\, (\mathbf{R}^K)^{\mathtt{t}} \left[\int_K \nabla\varphi_i^K\cdot\nabla\varphi_j^K\right] \mathbf{R}^K\right)$$

$$= \mathrm{kron}\left(\mathbf{I}_2,\, (\mathbf{R}^K)^{\mathtt{t}} \left[\begin{array}{c:c} 0 & \mathbf{0}_{1\times(m1-1)} \\ \hdashline \mathbf{0}_{(m1-1)\times 1} & \mathbf{M}_{\mathrm{grad},K} \end{array}\right] \mathbf{R}^K\right)$$

$$= \mathrm{kron}\left(\mathbf{I}_2,\, (\widehat{\mathbf{R}}_0)^{\mathtt{t}}\, \mathbf{M}_{\mathrm{grad},K}\, \widehat{\mathbf{R}}_0\right),$$

which $\widehat{\mathbf{R}}_0$ is defined in (3.10). Next, proceeding similar to $\mathbf{A}^K_{\text{sta}}$, we obtain that $\mathbf{D}^K_{\text{sta}} \in \mathbb{R}^{2n^V_k \times 2n^V_k}$ can be written as:

$$\mathbf{D}^K_{\text{sta}} := \left[\mathcal{S}^K_V\left(\vec{\boldsymbol{\psi}}^K_i - \mathcal{R}^K_k(\vec{\boldsymbol{\psi}}^K_i), \vec{\boldsymbol{\psi}}^K_j - \mathcal{R}^K_k(\vec{\boldsymbol{\psi}}^K_j)\right)\right] = \text{kron}(\mathbf{I}_2, \mathbf{H}^{\mathtt{t}}_D\, \mathbf{H}_D)\,,$$

where $\mathbf{H}_D \in \mathbb{R}^{n^V_k \times n^V_k}$ is given by:

$$\mathbf{H}_D := \mathbf{I}_{n^V_k} - \left[\begin{array}{c}\mathbf{P}_{\text{eval}} \\ \hdashline \mathbf{H}_3\end{array}\right]\mathbf{R}^K\,.$$

The matrix $\mathbf{H}_3 \in \mathbb{R}^{m_0 \times m_1}$ is the submatrix of $\mathbf{M}^{(k+1)}_{\text{mass},K}$, whose index range is $[1, m_0] \times [1, m_1]$. Furthermore, let $\mathbf{v}^v_1, \mathbf{v}^v_2, \ldots, \mathbf{v}^v_{d_K}$ be the vertices of K and, for each $e \in \partial K$, let $\mathbf{v}^e_1, \mathbf{v}^e_2, \ldots, \mathbf{v}^e_k$ be the k uniformly spaced points on e (sorted with the respective orientation), which define m^V_e (cf. (2.16)). Then, gathering all these points in a matrix $\mathbf{p} := [p_{ij}] \in \mathbb{R}^{2\times((k+1)d_K)}$ as follows:

$$\mathbf{p} := \left[\begin{array}{c:c:c:c:c}\mathbf{v}^v_1, \ldots, \mathbf{v}^v_{d_K} & \mathbf{v}^{e_1}_1, \ldots, \mathbf{v}^{e_1}_k & \mathbf{v}^{e_2}_1, \ldots, \mathbf{v}^{e_2}_k & \cdots & \mathbf{v}^{e_{d_K}}_1, \ldots, \mathbf{v}^{e_{d_K}}_k\end{array}\right],$$

we introduce the matrix $\mathbf{P}_{\text{eval}} := \left[\varphi^K_j(p_{1i},\ p_{2i})\right] \in \mathbb{R}^{((k+1)d_K)\times m_1}$, which completes the previous definition of $\mathbf{H}_D$.

On the other hand, the operator $\mathbf{D}^K_{\text{bou}} \in \mathbb{R}^{2n^V_k \times 2n^V_k}$ is given by

$$\mathbf{D}^K_{\text{bou}} := \left[\int_{\partial K \cap \Gamma} \vec{\boldsymbol{\psi}}^K_i \cdot \vec{\boldsymbol{\psi}}^K_j\right] = \text{kron}\left(\mathbf{I}_2, \sum_{e \in \partial K \cap \Gamma} \left[\int_e \psi^K_i\, \psi^K_j\right]\right),$$

where, employing (3.2), it follows that

$$\begin{aligned}\mathbf{D}^K_{\text{bou}} &= \text{kron}\left(\mathbf{I}_2, \sum_{e \in \partial K \cap \Gamma} (\mathbf{C}^e_L)^{\mathtt{t}} \left[\int_e \mathcal{L}^e_i\, \mathcal{L}^e_j\right] \mathbf{C}^e_L\right) \\ &= \text{kron}\left(\mathbf{I}_2, \sum_{e \in \partial K \cap \Gamma} h_e\, (\mathbf{C}^e_L)^{\mathtt{t}}\, \mathbf{M}_{\text{mass},L}\, \mathbf{C}^e_L\right).\end{aligned}$$

We end this section by considering the operator $\mathbf{b}^K_3 \in \mathbb{R}^{2n^V_k \times 1}$, which, employing (4.2), is given by:

$$\begin{aligned}\mathbf{b}^K_3 &:= \left[\int_{\partial K \cap \Gamma} \vec{\boldsymbol{\psi}}^K_i \cdot \mathbf{g}\right] = \sum_{e \in \partial K \cap \Gamma} \left[\begin{array}{c}\left[\int_e \psi^K_i\, g_1\right] \\ \hdashline \left[\int_e \psi^K_i\, g_2\right]\end{array}\right] \\ &= \sum_{e \in \partial K \cap \Gamma} \left[\begin{array}{c}(\mathbf{C}^e_L)^{\mathtt{t}} \left[\int_e \mathcal{L}^e_i\, \phi^e_j\right] \mathbf{p}^e_{g_\ell} 1 \\ \hdashline (\mathbf{C}^e_L)^{\mathtt{t}} \left[\int_e \mathcal{L}^e_i\, \phi^e_j\right] \mathbf{p}^e_{g_\ell} 2\end{array}\right] = \sum_{e \in \partial K \cap \Gamma} \left[\begin{array}{c}(\mathbf{C}^e_L)^{\mathtt{t}}\, \mathbf{M}_{\text{Lag},e}\, \mathbf{p}^e_{g_\ell} 1 \\ \hdashline (\mathbf{C}^e_L)^{\mathtt{t}}\, \mathbf{M}_{\text{Lag},e}\, \mathbf{p}^e_{g_\ell} 2\end{array}\right].\end{aligned}$$

Furthermore, using (4.3), we describe the implementation of the operator:

$$
\begin{aligned}
\mathbf{b}_4^K &:= \left[\int_K \vec{\boldsymbol{\psi}}_i^K \cdot \mathcal{P}_k^K(\mathbf{f})\right] = \left[\begin{array}{c} \left[\int_K P_k^K(\psi_i^K)\,\varphi_j^K\right]\mathbf{p}_{f_\ell}^K 1 \\ \left[\int_K P_k^K(\psi_i^K)\,\varphi_j^K\right]\mathbf{p}_{f_\ell}^K 2 \end{array}\right] \\
&= \left[\begin{array}{c} (\mathbf{P}_V)^{\mathrm{t}}\left[\int_K \varphi_i^K\,\varphi_j^K\right]\mathbf{p}_{f_\ell}^K 1 \\ \hdashline (\mathbf{P}_V)^{\mathrm{t}}\left[\int_K \varphi_i^K\,\varphi_j^K\right]\mathbf{p}_{f_\ell}^K 2 \end{array}\right] = \left[\begin{array}{c} (\mathbf{P}_V)^{\mathrm{t}}\,\mathbf{M}_{\mathrm{mass},K}\,\mathbf{p}_{f_\ell}^K 1 \\ \hdashline (\mathbf{P}_V)^{\mathrm{t}}\,\mathbf{M}_{\mathrm{mass},K}\,\mathbf{p}_{f_\ell}^K 2 \end{array}\right] \\
&= \left[\begin{array}{c} (\mathbf{P}_V)^{\mathrm{t}}\left[\int_K \varphi_i^K\, f_1\right] \\ \hdashline (\mathbf{P}_V)^{\mathrm{t}}\left[\int_K \varphi_i^K\, f_2\right] \end{array}\right] \in \mathbb{R}^{2n_k^V \times 1}.
\end{aligned}
$$

4.3 Operators related to the elements of $\mathbf{H}_k^K$ and $\mathbf{V}_k^K$

Next, we define the following operators:

- $\mathbf{B}^K := \left[\int_K \mathbf{div}(\vec{\boldsymbol{\Psi}}_i^K)\cdot\vec{\boldsymbol{\psi}}_j^K\right] \in \mathbb{R}^{2n_k^H \times 2n_k^V}$
- $\mathbf{C}^K := \left[\int_K \boldsymbol{\mathcal{P}}_k^K(\nabla\vec{\boldsymbol{\psi}}_i^K) : \left[\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\right]^{\mathrm{d}}\right] \in \mathbb{R}^{2n_k^V \times 2n_k^H}$

where, using the matrices $\mathbf{P}_V \in \mathbb{R}^{m \times n_k^V}$ and $\mathcal{D}\mathbf{P}_V \in \mathbb{R}^{(4m)\times(2n_k^V)}$ defined in Section 3.4, we can assemble $\mathbf{B}^K \in \mathbb{R}^{2n_k^H \times 2n_k^V}$ and $\mathbf{C}^K \in \mathbb{R}^{2n_k^V \times 2n_k^H}$ similar to previous operators. Indeed, we have that

$$
\begin{aligned}
\mathbf{B}^K &:= \left[\int_K \mathbf{div}(\vec{\boldsymbol{\Psi}}_i^K)\cdot\vec{\boldsymbol{\psi}}_j^K\right] = \operatorname{kron}\left(\mathbf{I}_2,\, \left[\int_K \operatorname{div}(\Psi_i^K)\,\psi_j^K\right]\right) \\
&= \operatorname{kron}\left(\mathbf{I}_2,\, (\mathbf{M}_{\mathrm{div}})^{\mathrm{t}}\left[\int_K \varphi_i^K\,\psi_j^K\right]\right) = \operatorname{kron}\left(\mathbf{I}_2,\, (\mathbf{M}_{\mathrm{div}})^{\mathrm{t}}\left[\int_K \varphi_i^K\,P_k^K(\psi_j^K)\right]\right) \\
&= \operatorname{kron}\left(\mathbf{I}_2,\, (\mathbf{M}_{\mathrm{div}})^{\mathrm{t}}\left[\int_K \varphi_i^K\,\varphi_j^K\right]\mathbf{P}_V\right) = \operatorname{kron}\left(\mathbf{I}_2,\, (\mathbf{M}_{\mathrm{div}})^{\mathrm{t}}\,\mathbf{M}_{\mathrm{mass},K}\,\mathbf{P}_V\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{C}^K &:= \left[\int_K \boldsymbol{\mathcal{P}}_k^K(\nabla\vec{\boldsymbol{\psi}}_i^K) : \left[\boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_j^K)\right]^{\mathrm{d}}\right] = (\mathcal{D}\mathbf{P}_V)^{\mathrm{t}}\left[\int_K \Phi_i^K : (\Phi_j^K)^{\mathrm{d}}\right]\mathbf{P}^K \\
&= (\mathcal{D}\mathbf{P}_V)^{\mathrm{t}}\operatorname{kron}\left(\mathbf{M}_{\mathrm{dev}},\, \mathbf{M}_{\mathrm{mass},K}\right)\mathbf{P}^K,
\end{aligned}
$$

where the matrix $\mathbf{M}_{\mathrm{dev}} \in \mathbb{R}^{4\times 4}$ is defined in (4.1).

5 A particular example: the Navier-Stokes problem

In this section, we show how to use the previously defined discrete operators in a particular formulation. More precisely, we present some specific aspects on the computational implementation of a mixed virtual element method for the two-dimensional pseudostress-velocity formulation of the Navier-Stokes equations with Dirichlet boundary conditions. Indeed, the formulation used below was originally proposed and analyzed in [8]. Here, we recall the continuous and discrete formulations, and propose an algorithm for the assembly of the associated global linear system for Newton's iteration. Finally, a numerical example illustrating the performance of the mixed-VEM scheme and confirming these theoretical rates is presented.

5.1 The continuous problem

We begin, recalling from [8, Section 2], the boundary value problem of interest. Indeed, letting $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary Γ, we consider the stationary Navier-Stokes equations with nonhomogeneous Dirichlet boundary conditions. More precisely, given a volume force $\mathbf{f} \in \mathbf{L}(\Omega)$ and a Dirichlet datum $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we seek a vector field (the velocity) $\mathbf{u}$ and a scalar field (the pressure) p of a fluid on Ω, such that

$$\begin{gathered} -\mu\Delta\mathbf{u} + (\nabla\mathbf{u})\,\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega\,, \qquad \operatorname{div}(\mathbf{u}) = 0 \text{ in } \Omega\,, \\ \mathbf{u} = \mathbf{g} \text{ on } \Gamma\,, \text{ and } \int_\Omega p = 0\,, \end{gathered} \tag{5.1}$$

where $\mu > 0$ is the viscosity constant. In addition, it is important to recall here that the incompressibility condition given by the second equation of (5.1), establishes that the datum $\mathbf{g}$ satisfies the compatibility condition $\int_\Gamma \mathbf{g}\cdot\boldsymbol{n} = 0$, where $\boldsymbol{n}$ stands for the unit outward normal at Γ.

On the other hand, defining the constant $c := -\frac{1}{2|\Omega|}\|\mathbf{u}\|^2_{0,\Omega}$ and $\mathbf{I}_2$ the identity matrix of $\mathbb{R}^{2\times 2}$, we introduce the pseudostress tensor (see [30–32])

$$\boldsymbol{\sigma} := \mu\nabla\mathbf{u} - \mathbf{u}\otimes\mathbf{u} - (p+c)\,\mathbf{I}_2 \text{ in } \Omega\,, \tag{5.2}$$

which allows us to arrive at the equivalent system: Find the pseudostress $\boldsymbol{\sigma}$ and the velocity $\mathbf{u}$ such that

$$\begin{gathered} \boldsymbol{\sigma}^{\mathrm{d}} = \mu\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \text{ in } \Omega - \mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} \text{ in } \Omega\,, \\ \mathbf{u} = \mathbf{g} \text{ on } \Gamma\,, \text{ and } \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\,, \end{gathered} \tag{5.3}$$

where the pseudostress variable has eliminated the pressure from the original model (5.1), which can be recover through the postprocessing formula:

$$p = -\frac{1}{2}\left\{\operatorname{tr}(\boldsymbol{\sigma}) + \operatorname{tr}(\mathbf{u}\otimes\mathbf{u})\right\} - c \text{ in } \Omega\,,$$

which is obtained from (5.2) and the incompressibility condition.

Next, we let $\mathbb{H}_0(\mathbf{div};\Omega) := \big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega) \,:\, \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\big\}$, and recall from [8, Section 2] the following redundant terms:

$$\begin{aligned}
\kappa_1\int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) &= -\kappa_1\int_\Omega \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega)\,,\\
\kappa_2\int_\Omega \Big\{\mu\nabla\mathbf{u} - \boldsymbol{\sigma}^{\mathrm{d}} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}\Big\} : \nabla\mathbf{v} &= 0 \qquad\qquad\qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\,,\\
\kappa_3\int_\Gamma \mathbf{u}\cdot\mathbf{v} &= \kappa_3\int_\Gamma \mathbf{g}\cdot\mathbf{v} \qquad\quad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\,,
\end{aligned}$$

where, according to [8, Theorem 2.1], the parameters $\kappa_1,\kappa_2,\kappa_3$ must satisfy that $\kappa_1,\kappa_3 > 0$ and $0 < \kappa_2 < 2\mu$. Then, we consider the continuous formulation of (5.3) introduced in [8, Section 2], whose well-posedness has been established in [8, Theorem 2.1]. More precisely, we seek $\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{X} := \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ such that

$$a(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) + b(\mathbf{u};\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) = F(\vec{\boldsymbol{\tau}}) \qquad \forall\,\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}\,, \tag{5.4}$$

where $a : \mathbb{X}\times\mathbb{X}\to\mathbb{R}$ is the bilinear form

$$\begin{aligned}
a(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) := &\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \kappa_1\int_\Omega \mathbf{div}(\boldsymbol{\zeta})\cdot\mathbf{div}(\boldsymbol{\tau}) + \kappa_2\mu\int_\Omega \nabla\mathbf{w}:\nabla\mathbf{v} + \kappa_3\int_\Gamma \mathbf{w}\cdot\mathbf{v}\\
&- \mu\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\zeta}) + \mu\int_\Omega \mathbf{w}\cdot\mathbf{div}(\boldsymbol{\tau}) - \kappa_2\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \nabla\mathbf{v}
\end{aligned}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta},\mathbf{w}),\ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}$, $F : \mathbb{X}\to\mathbb{R}$ is the linear functional

$$F(\vec{\boldsymbol{\tau}}) := \mu\langle\boldsymbol{\tau}\boldsymbol{n},\mathbf{g}\rangle_\Gamma - \kappa_1\int_\Omega \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) + \mu\int_\Omega \mathbf{f}\cdot\mathbf{v} + \kappa_3\int_\Gamma \mathbf{g}\cdot\mathbf{v}\,,$$

for all $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}$, and given $\mathbf{z}\in\mathbf{H}^1(\Omega)$, $b(\mathbf{z};\cdot,\cdot) : \mathbb{X}\times\mathbb{X}\to\mathbb{R}$ is the bilinear form

$$b(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) := \int_\Omega (\mathbf{w}\otimes\mathbf{z})^{\mathrm{d}} : \big\{\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}\big\}\,,$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta},\mathbf{w}),\ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}$.

5.2 The mixed-VEM formulation

Now, given an integer $k\geq 0$, we consider the virtual element subspace $\mathbb{X}_h$ of $\mathbb{X} := \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ given by

$$\mathbb{X}_h := \Big\{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X} : \quad \boldsymbol{\tau}|_K\in\mathbb{H}_k^K \ \text{ and } \ \mathbf{v}|_K\in\mathbf{V}_k^K \quad \forall\,K\in\mathcal{T}_h\Big\}\,, \tag{5.5}$$

where, for each $K\in\mathcal{T}_h$, the virtual subspaces $\mathbb{H}_k^K$ and $\mathbf{V}_k^K$ are defined in (2.6) and (2.18), respectively. In turn, we now aim to define the nonlinear mixed virtual element scheme associated with (5.4) and introduced in [8, Section 5]. That is, we seek $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{X}_h$ such that

$$a_h(\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) + b_h(\mathbf{u}_h;\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) = F_h(\vec{\boldsymbol{\tau}}_h) \qquad \forall\,\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_h\,. \tag{5.6}$$

Equation (5.6) will be one of the few expression where we use the subscript h for the elements in $\mathbb{X}_h$. In what follows we mostly omit that subscript in order to simplify the notation. Thus, we have that $a_h : \mathbb{X}_h \times \mathbb{X}_h \to \mathbb{R}$ is the bilinear form defined by:

$$\begin{aligned} a_h(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) := \sum_{K\in\mathcal{T}_h} \Bigg\{ & \int_K \big[\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big]^{\mathrm{d}} : \big[\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\big]^{\mathrm{d}} \,+\, \mathcal{S}_H^K\big(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\big) \\ & + \kappa_1 \int_K \mathbf{div}(\boldsymbol{\zeta}) \cdot \mathbf{div}(\boldsymbol{\tau}) \,+\, \mu \int_K \mathbf{w} \cdot \mathbf{div}(\boldsymbol{\tau}) \,-\, \mu \int_K \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\zeta}) \\ & - \kappa_2 \int_K \big[\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big]^{\mathrm{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v}) \,+\, \kappa_2\mu \int_K \nabla\mathcal{R}_k^K(\mathbf{w}) : \nabla\mathcal{R}_k^K(\mathbf{v}) \\ & + \mathcal{S}_V^K\big(\mathbf{w} - \mathcal{R}_k^K(\mathbf{w}), \mathbf{v} - \mathcal{R}_k^K(\mathbf{v})\big) \,+\, \kappa_3 \int_{\partial K\cap\Gamma} \mathbf{w}\cdot\mathbf{v} \Bigg\} \end{aligned}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_h$, $F_h : \mathbb{X}_h \to \mathbb{R}$ is the linear functional

$$F_h(\vec{\boldsymbol{\tau}}) := \sum_{K\in\mathcal{T}_h} \left\{ \mu \int_{\partial K\cap\Gamma} \boldsymbol{\tau}\boldsymbol{n}\cdot\mathbf{g} \,-\, \kappa_1 \int_K \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \kappa_3 \int_{\partial K\cap\Gamma} \mathbf{g}\cdot\mathbf{v} \,+\, \mu \int_K \mathcal{P}_k^K(\mathbf{f})\cdot\mathbf{v} \right\}$$

for all $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_h$, and given $\mathbf{z} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{z}|_K \in \mathbf{V}_k^K$ for all $K \in \mathcal{T}_h$, the bilinear form $b_h(\mathbf{z}; \cdot, \cdot) : \mathbb{X}_h \times \mathbb{X}_h \to \mathbb{R}$ is given by

$$b_h(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) := \sum_{K\in\mathcal{T}_h} \left\{ \int_K \big[\mathcal{P}_k^K(\mathbf{w}) \otimes \mathcal{P}_k^K(\mathbf{z})\big]^{\mathrm{d}} : \big[\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) - \kappa_2\boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v})\big] \right\} \tag{5.7}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_h$.

The bilinear forms $\mathcal{S}_H^K$ and $\mathcal{S}_V^K$ are defined in (2.9) and (2.19), respectively. Moreover, we recall here that $\mathcal{P}_k^K : \mathbf{L}(K) \to \mathbf{P}_k(K)$ and $\boldsymbol{\mathcal{P}}_k^K : \mathbb{L}(K) \to \mathbb{P}_k(K)$ are the corresponding $\mathbf{L}(K)$ and $\mathbb{L}(K)$ orthogonal projections (see at the end of Section 2.1). On the other hand, under suitable assumptions, the discrete scheme (5.6) has a unique solution, which was proved in [8, Theorem 5.1], whereas in [8, Theorem 5.3], the respective a priori error estimates were established.

5.3 Newton's iteration and the linear system assembly

In this section, Newton's method is described as an option to solve the discrete scheme (5.6), which as usual requires the assembly and resolution of a series of linear systems. Thus, we now aim to propose the following Newton's iteration for the discrete scheme (5.6): Given $\vec{\boldsymbol{\sigma}}_h^{(0)} := (\boldsymbol{\sigma}_h^{(0)}, \mathbf{u}_h^{(0)}) \in \widetilde{\mathbb{X}}_h$ and $\xi_h^{(0)} \in \mathbb{R}$, for each integer $s \geq 0$, we apply the iteration:

1. Find $\vec{\boldsymbol{\zeta}}_h^{(s)} := (\boldsymbol{\zeta}_h^{(s)}, \mathbf{w}_h^{(s)}) \in \widetilde{\mathbb{X}}_h$ and $\eta_h^{(s)} \in \mathbb{R}$ such that

$$\begin{aligned} a_h(\vec{\boldsymbol{\zeta}}_h^{(s)}, \vec{\boldsymbol{\tau}}_h) \,+\, \mathcal{D}b_h(\mathbf{u}_h^{(s)}; \vec{\boldsymbol{\zeta}}_h^{(s)}, \vec{\boldsymbol{\tau}}_h) + \eta_h^{(s)} \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) \;&=\; F_h(\vec{\boldsymbol{\tau}}_h) \,-\, a_h(\vec{\boldsymbol{\sigma}}_h^{(s)}, \vec{\boldsymbol{\tau}}_h) \\ &\quad\; - b_h(\mathbf{u}_h^{(s)}; \vec{\boldsymbol{\sigma}}_h^{(s)}, \vec{\boldsymbol{\tau}}_h) \\ &\quad\; - \xi_h^{(s)} \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h)\,, \\ \lambda_h \int_\Omega \operatorname{tr}(\boldsymbol{\zeta}_h^{(s)}) \;&=\; -\lambda_h \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h^{(s)})\,, \end{aligned} \tag{5.8}$$

   for all $\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \widetilde{\mathbb{X}}_h$ and for all $\lambda_h \in \mathbb{R}$, where $\vec{\boldsymbol{\sigma}}_h^{(s)} := (\boldsymbol{\sigma}_h^{(s)}, \mathbf{u}_h^{(s)}) \in \widetilde{\mathbb{X}}_h$.
2. Compute $\vec{\boldsymbol{\sigma}}_h^{(s+1)} := \vec{\boldsymbol{\sigma}}_h^{(s)} + \vec{\boldsymbol{\zeta}}_h^{(s)}$ and $\xi_h^{(s+1)} := \xi_h^{(s)} + \eta_h^{(s)}$.

Here, for each $\mathbf{z} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{z}|_K \in \mathbf{V}_k^K$ for all $K \in \mathcal{T}_h$, the bilinear form $\mathcal{D}b_h(\mathbf{z};\,\cdot\,,\,\cdot\,) : \widetilde{\mathbb{X}}_h \times \widetilde{\mathbb{X}}_h \to \mathbb{R}$ is the Gâteaux derivative of $b_h(\mathbf{z};\,\cdot\,,\,\cdot\,)$ (cf. (5.7)), given by

$$\mathcal{D}b_h(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) := \sum_{K \in \mathcal{T}_h} \left\{ \int_K \left[\mathcal{P}_k^K(\mathbf{w}) \otimes \mathcal{P}_k^K(\mathbf{z}) + \mathcal{P}_k^K(\mathbf{z}) \otimes \mathcal{P}_k^K(\mathbf{w})\right]^{\mathrm{d}} : \left[\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) - \kappa_2 \boldsymbol{\mathcal{P}}_k^K(\nabla \mathbf{v})\right] \right\},$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w})$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \widetilde{\mathbb{X}}_h$. Furthermore, in order to relax the restrictions of $\mathbb{X}_h$ (cf. (5.5)), $\xi \in \mathbb{R}$ is introduced as the Lagrange multiplier that allows us to extract the condition $\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h^{(s)}) = 0$ (see, e.g., [8, eq. (5.3)]). Then, we replace the space $\mathbb{X}_h$ as follows:

$$\widetilde{\mathbb{X}}_h \;:=\; \Big\{ (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega) \;:\quad \boldsymbol{\tau}|_K \in \mathbb{H}_k^K \;\text{ and }\; \mathbf{v}|_K \in \mathbf{V}_k^K \quad \forall\, K \in \mathcal{T}_h \Big\},$$

which is identical to $\mathbb{X}_h$ (cf. (5.5)), except that the null trace integral condition is no longer imposed on its elements.

Next, the global linear system associated to (5.8) has the matrix structure:

$$\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)} \left[\begin{array}{c} \boldsymbol{\zeta}_h^{(s)} \\ \hdashline \mathbf{w}_h^{(s)} \\ \hdashline \eta_h^{(s)} \end{array}\right] \;=\; \boldsymbol{b} \,-\, \boldsymbol{\mathcal{A}}^{(s)} \left[\begin{array}{c} \boldsymbol{\sigma}_h^{(s)} \\ \hdashline \mathbf{u}_h^{(s)} \\ \hdashline \xi_h^{(s)} \end{array}\right], \tag{5.9}$$

where $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}, \boldsymbol{\mathcal{A}}^{(s)} \in \mathbb{R}^{N\times N}$, and $\boldsymbol{b} \in \mathbb{R}^{N\times 1}$, with s indicating the dependence of $\mathbf{u}_h^{(s)}$. In addition, N is the size of the system (5.9) which is given by (see (2.4) and (2.16)):

$$\begin{aligned} N := {} & 2\cdot \underbrace{(k+1)}_{m^H_{q,\boldsymbol{n}}} \cdot(\text{\# of edges in } \mathcal{T}_h) \;+\; 2\cdot\Bigg(\underbrace{\frac{(k+1)(k+2)}{2} - 1}_{m^H_{q,\mathrm{div}}} + \underbrace{\frac{k(k+1)}{2}}_{m^H_{\mathbf{q},\mathrm{rot}}} \Bigg)\cdot(\text{\# of elements in } \mathcal{T}_h) \\ & + 2\cdot \underbrace{1}_{m^V_{i,v}} \cdot(\text{\# of nodes in } \mathcal{T}_h) \;+\; 2\cdot \underbrace{k}_{m^V_e} \cdot(\text{\# of edges in } \mathcal{T}_h) \\ & + 2\cdot\Bigg(\underbrace{\frac{k(k+1)}{2}}_{m^V_{q,K}} \Bigg)\cdot(\text{\# of elements in } \mathcal{T}_h) \;+\; \underbrace{1}_{\xi}, \end{aligned} \tag{5.10}$$

which indicates that there is N unknowns associated with the degrees of freedom.

On the other hand, as is usual in finite element methods, the explicit construction of the coefficient matrix and the right-hand side vector in the system (5.9), is done by assembling local discrete operators from each element $K \in \mathcal{T}_h$. More precisely, for each $K \in \mathcal{T}_h$, consider the local version (i.e., the contribution of K) of the system (5.9) given by:

$$\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}\big|_K \;=\; \left[\begin{array}{c:c:c} \mathbf{A}^K_{\mathrm{dev}} + \mathbf{A}^K_{\mathrm{sta}} + \kappa_1\,\mathbf{A}^K_{\mathrm{div}} & \mu\,\mathbf{B}^K + \mathcal{D}\mathbf{G}^{(s)}_{1,K} & \mathbf{a}^K_{\mathrm{tra}} \\ \hdashline -\mu\,(\mathbf{B}^K)^{\mathrm{t}} - \kappa_2\,\mathbf{C}^K & \kappa_2\mu\,\mathbf{D}^K_{\mathrm{gra}} + \mathbf{D}^K_{\mathrm{sta}} + \kappa_3\,\mathbf{D}^K_{\mathrm{bou}} + \mathcal{D}\mathbf{G}^{(s)}_{2,K} & \mathbf{0} \\ \hdashline (\mathbf{a}^K_{\mathrm{tra}})^{\mathrm{t}} & \mathbf{0} & 0 \end{array}\right],$$

$$\boldsymbol{\mathcal{A}}^{(s)}\big|_K \;=\; \left[\begin{array}{c:c:c} \mathbf{A}^K_{\mathrm{dev}} + \mathbf{A}^K_{\mathrm{sta}} + \kappa_1\,\mathbf{A}^K_{\mathrm{div}} & \mu\,\mathbf{B}^K + \mathbf{G}^{(s)}_{1,K} & \mathbf{a}^K_{\mathrm{tra}} \\ \hdashline -\mu\,(\mathbf{B}^K)^{\mathrm{t}} - \kappa_2\,\mathbf{C}^K & \kappa_2\mu\,\mathbf{D}^K_{\mathrm{gra}} + \mathbf{D}^K_{\mathrm{sta}} + \kappa_3\,\mathbf{D}^K_{\mathrm{bou}} + \mathbf{G}^{(s)}_{2,K} & \mathbf{0} \\ \hdashline (\mathbf{a}^K_{\mathrm{tra}})^{\mathrm{t}} & \mathbf{0} & 0 \end{array}\right],$$

and

$$\boldsymbol{b}\big|_K \;=\; \left[\begin{array}{c} \mu\,\mathbf{b}^K_1 - \kappa_1\,\mathbf{b}^K_2 \\ \hdashline \kappa_3\,\mathbf{b}^K_3 + \mu\,\mathbf{b}^K_4 \\ \hdashline 0 \end{array}\right].$$

The explicit construction of the discrete operators $\mathbf{A}^K_{\mathrm{dev}}$, $\mathbf{A}^K_{\mathrm{sta}}$, $\mathbf{A}^K_{\mathrm{div}}$, $\mathbf{B}^K$, $\mathbf{a}^K_{\mathrm{tra}}$, $\mathbf{D}^K_{\mathrm{gra}}$, $\mathbf{D}^K_{\mathrm{sta}}$, $\mathbf{D}^K_{\mathrm{bou}}$, $\mathbf{C}^K$, $\mathbf{b}^K_1$, $\mathbf{b}^K_2$, $\mathbf{b}^K_3$, and $\mathbf{b}^K_4$ are detailed in Section 4. Conversely, the operators:

- $\mathbf{G}^{(s)}_{1,K} \;:=\; \left[\displaystyle\int_K \boldsymbol{\mathcal{P}}^K_k(\vec{\boldsymbol{\Psi}}^K_i) : \big[\mathcal{P}^K_k(\vec{\boldsymbol{\psi}}^K_j) \otimes \mathcal{P}^K_k(\mathbf{u}^{(s)}_h)\big]^{\mathrm{d}}\right] \in \mathbb{R}^{2n^H_k \times 2n^V_k}$
- $\mathbf{G}^{(s)}_{2,K} \;:=\; \left[-\kappa_2 \displaystyle\int_K \boldsymbol{\mathcal{P}}^K_k(\nabla\vec{\boldsymbol{\psi}}^K_i) : \big[\mathcal{P}^K_k(\vec{\boldsymbol{\psi}}^K_j) \otimes \mathcal{P}^K_k(\mathbf{u}^{(s)}_h)\big]^{\mathrm{d}}\right] \in \mathbb{R}^{2n^V_k \times 2n^V_k}$
- $\mathcal{D}\mathbf{G}^{(s)}_{1,K} \;:=\; \left[\displaystyle\int_K \boldsymbol{\mathcal{P}}^K_k(\vec{\boldsymbol{\Psi}}^K_i) : \big[\mathcal{P}^K_k(\vec{\boldsymbol{\psi}}^K_j) \otimes \mathcal{P}^K_k(\mathbf{u}^{(s)}_h) + \mathcal{P}^K_k(\mathbf{u}^{(s)}_h) \otimes \mathcal{P}^K_k(\vec{\boldsymbol{\psi}}^K_j)\big]^{\mathrm{d}}\right] \in \mathbb{R}^{2n^H_k \times 2n^V_k}$
- $\mathcal{D}\mathbf{G}^{(s)}_{2,K} := \left[-\kappa_2 \displaystyle\int_K \boldsymbol{\mathcal{P}}^K_k(\nabla\vec{\boldsymbol{\psi}}^K_i) : \big[\mathcal{P}^K_k(\vec{\boldsymbol{\psi}}^K_j) \otimes \mathcal{P}^K_k(\mathbf{u}^{(s)}_h) + \mathcal{P}^K_k(\mathbf{u}^{(s)}_h) \otimes \mathcal{P}^K_k(\vec{\boldsymbol{\psi}}^K_j)\big]^{\mathrm{d}}\right] \in \mathbb{R}^{2n^V_k \times 2n^V_k}$

will be defined in Section 5.4. At the moment, for each $K \in \mathcal{T}_h$, we assume that these were already calculated, in order to describe below the assembly of the global linear system (5.9).

According to the previous discussion, the global matrices $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{\mathcal{A}}^{(s)} \in \mathbb{R}^{N\times N}$, along with the vector $\boldsymbol{b} \in \mathbb{R}^{N\times 1}$, can be assembled through the following algorithm:

```
Define $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)} := \mathbf{0}$, $\boldsymbol{\mathcal{A}}^{(s)} := \mathbf{0}$, and $\boldsymbol{b} := \mathbf{0}$
Define $m^H := \frac{1}{2}(k+1)(k+2) - 1 + \frac{1}{2}k(k+1)$, and $m^V := \frac{1}{2}k(k+1)$
Define $w_0 :=$ (# of edges in $\mathcal{T}_h$) $\cdot\, 2(k+1)$ + (# of elements in $\mathcal{T}_h$) $\cdot\, 2m^H$
For each $K \in \mathcal{T}_h$ do:
    Construct the discrete operators: $\mathbf{A}^K_{\mathrm{dev}}$, $\mathbf{A}^K_{\mathrm{sta}}$, $\mathbf{A}^K_{\mathrm{div}}$, $\mathbf{B}^K$, $\mathbf{a}^K_{\mathrm{tra}}$, $\mathbf{D}^K_{\mathrm{gra}}$, $\mathbf{D}^K_{\mathrm{sta}}$, $\mathbf{D}^K_{\mathrm{bou}}$, $\mathbf{C}^K$, $\mathbf{G}^{(s)}_{1,K}$, $\mathbf{G}^{(s)}_{2,K}$, $\mathcal{D}\mathbf{G}^{(s)}_{1,K}$, $\mathcal{D}\mathbf{G}^{(s)}_{2,K}$, $\mathbf{b}^K_1$, $\mathbf{b}^K_2$, $\mathbf{b}^K_3$, and $\mathbf{b}^K_4$
    Define $n^H := (k+1)d_K + m^H$, and $n^V := (k+1)d_K + m^V$
    Define $\mathbf{p}^H := (p^H_i) \in \mathbb{R}^{n^H\times 1}$, and $\mathbf{q}^H := (q^H_i) \in \mathbb{R}^{n^H\times 1}$
    Define $\mathbf{p}^V := (p^V_i) \in \mathbb{R}^{n^V\times 1}$, and $\mathbf{q}^V := (q^V_i) \in \mathbb{R}^{n^V\times 1}$
    For $e = 1$ until $d_K$ do:
        Let $I_e$ be the global index of the local edge $e$ in $\mathcal{T}_h$
        Define $w := (I_e - 1)\cdot 2(k+1)$
        For $r = 1$ until $k+1$ do:
            Define $i := (e-1)(k+1) + r$
            Set $p^H_i := w + r$
            Set $q^H_i := w + (k+1) + r$
        End for of $r$
        Let $I_n$ be the global index of the $e$th local node in $\mathcal{T}_h$
        Set $p^V_e := w_0 + 2(I_n - 1) + 1$
        Set $q^V_e := w_0 + 2(I_n - 1) + 2$
        Define $w := w_0$ + (# of nodes in $\mathcal{T}_h$) $\cdot\, 2$ + $(I_e - 1)\cdot 2k$
        If edge $e$ has positive orientation in $\mathcal{T}_h$ do:
            For $r = 1$ until $k$ do:
                Define $i := d_K + (e-1)\cdot k + r$
                Set $p^V_i := w + r$
                Set $q^V_i := w + k + r$
            End for of $r$
        Else
            For $r = 1$ until $k$ do:
                Define $i := d_K + (e-1)\cdot k + r$
                Set $p^V_i := w + (k+1-r)$
                Set $q^V_i := w + k + (k+1-r)$
            End for of $r$
        End if
    End for of $e$
    Let $I_K$ be the global index of the element $K$ in $\mathcal{T}_h$
    Define $w^H :=$ (# of edges in $\mathcal{T}_h$) $\cdot\, 2(k+1)$ + $(I_K - 1)\cdot 2m^H$
    For $r = 1$ until $m^H$ do:
```

```
38.         Define $i := (k+1)d_K + r$
39.         Set $p_i^H := w^H + r$
40.         Set $q_i^H := w^H + m^H + r$
41.     End for of $r$
42.     Define $w^V := w_0 + (\#\text{ of nodes in } \mathcal{T}_h)\cdot 2 + (\#\text{ of edges in } \mathcal{T}_h)\cdot 2k + (I_K - 1)\cdot 2m^V$
43.     For $r = 1$ until $m^V$ do:
44.         Define $i := (k+1)d_K + r$
45.         Set $p_i^V := w^V + r$
46.         Set $q_i^V := w^V + m^V + r$
47.     End for of $r$
48.     Define $\mathbf{u}^H := \begin{pmatrix} \mathbf{p}^H \\ \mathbf{q}^H \end{pmatrix} \in \mathbb{R}^{2n^H\times 1}$, and $\mathbf{u}^V := \begin{pmatrix} \mathbf{p}^V \\ \mathbf{q}^V \end{pmatrix} \in \mathbb{R}^{2n^V\times 1}$
49.     $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^H) := \mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^H) + \mathbf{A}_{\text{dev}}^K + \mathbf{A}_{\text{sta}}^K + \kappa_1\,\mathbf{A}_{\text{div}}^K$
50.     $\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^H) := \boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^H) + \mathbf{A}_{\text{dev}}^K + \mathbf{A}_{\text{sta}}^K + \kappa_1\,\mathbf{A}_{\text{div}}^K$
51.     $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^V) := \mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^V) + \mu\,\mathbf{B}^K + \mathcal{D}\mathbf{G}_{1,K}^{(s)}$
52.     $\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^V) := \boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,\mathbf{u}^V) + \mu\,\mathbf{B}^K + \mathbf{G}_{1,K}^{(s)}$
53.     $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,N) := \mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,N) + \mathbf{a}_{\text{tra}}^K$
54.     $\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,N) := \boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^H,N) + \mathbf{a}_{\text{tra}}^K$
55.     $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^H) := \mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^H) - \mu\,(\mathbf{B}^K)^{\mathtt{t}} - \kappa_2\,\mathbf{C}^K$
56.     $\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^H) := \boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^H) - \mu\,(\mathbf{B}^K)^{\mathtt{t}} - \kappa_2\,\mathbf{C}^K$
57.     $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^V) := \mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^V) + \kappa_2\mu\,\mathbf{D}_{\text{gra}}^K + \mathbf{D}_{\text{sta}}^K + \kappa_3\,\mathbf{D}_{\text{bou}}^K + \mathcal{D}\mathbf{G}_{2,K}^{(s)}$
58.     $\boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^V) := \boldsymbol{\mathcal{A}}^{(s)}(\mathbf{u}^V,\mathbf{u}^V) + \kappa_2\mu\,\mathbf{D}_{\text{gra}}^K + \mathbf{D}_{\text{sta}}^K + \kappa_3\,\mathbf{D}_{\text{bou}}^K + \mathbf{G}_{2,K}^{(s)}$
59.     $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(N,\mathbf{u}^H) := \mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}(N,\mathbf{u}^H) + (\mathbf{a}_{\text{tra}}^K)^{\mathtt{t}}$
60.     $\boldsymbol{\mathcal{A}}^{(s)}(N,\mathbf{u}^H) := \boldsymbol{\mathcal{A}}^{(s)}(N,\mathbf{u}^H) + (\mathbf{a}_{\text{tra}}^K)^{\mathtt{t}}$
61.     $\boldsymbol{b}(\mathbf{u}^H) := \boldsymbol{b}(\mathbf{u}^H) + \mu\,\mathbf{b}_1^K - \kappa_1\,\mathbf{b}_2^K$
62.     $\boldsymbol{b}(\mathbf{u}^V) := \boldsymbol{b}(\mathbf{u}^V) + \kappa_3\,\mathbf{b}_3^K + \mu\,\mathbf{b}_4^K$
63. End for of $K$
```

Regarding the procedure described previously, it is important to recall that the construction of discrete operators of line 5 is described in Sections 4 and 5.4. On the other hand, in lines from 7 to 48, we construct vectors $\mathbf{u}^H$ and $\mathbf{u}^V$, in order to map the local degrees of freedom of K to their corresponding location in the global system (5.9). In fact, $\mathbf{p}^H$ maps the first row of the tensor $\boldsymbol{\sigma}_h$, whereas $\mathbf{q}^H$ maps the second one. Similarly, $\mathbf{p}^V$ maps the first component of the vector $\mathbf{u}_h$ and $\mathbf{q}^H$ maps its second component (see Fig. 1). Furthermore, the global assembly is performed in lines 49 through 62, where, in particular, the notation $\boldsymbol{\mathcal{A}}(\mathbf{i},\mathbf{j})$ is related to access of the block of $\boldsymbol{\mathcal{A}}$ obtained through rows with indexes stored in $\mathbf{i}$ and columns with indexes stored in $\mathbf{j}$. In particular, we remark here that MATLAB allows us to perform these operations in natural way. Finally, observe that only the matrices $\mathcal{D}\mathbf{G}_1^{(s)}$, $\mathcal{D}\mathbf{G}_2^{(s)}$, $\mathbf{G}_1^{(s)}$, and $\mathbf{G}_2^{(s)}$ depend on the previous approximation and they are the main difference between the matrices $\mathcal{D}\boldsymbol{\mathcal{A}}^{(s)}$ and $\boldsymbol{\mathcal{A}}^{(s)}$, which can be used to improve the efficiency of the foregoing assembly.

5.4 The operators $\mathbf{G}^{(s)}_{1,K}$, $\mathbf{G}^{(s)}_{2,K}$, $\mathcal{D}\mathbf{G}^{(s)}_{1,K}$, and $\mathcal{D}\mathbf{G}^{(s)}_{2,K}$

We now aim to describe the implementation of the last four operators related to the nonlinearity of our problem. To do that, let $s \geq 0$ be an integer representing the current iteration of Newton's method. In turn, let $\mathbf{u}^{(s)}_{h,K} \in \mathbf{V}^K_k$ be the local approximation of $\mathbf{u}$ in the sth iteration of Newton's method, which satisfies that

$$\mathbf{u}^{(s)}_{h,K} \;:=\; \mathbf{u}^{(s)}_h\big|_K \;=\; \sum_{i=1}^{2n^V_k} \beta^K_i \, \vec{\boldsymbol{\psi}}^K_i \,, \tag{5.11}$$

where $\{\beta^K_i\}_{i=1}^{2n^V_k}$ are the respective local degrees of freedom of $\mathbf{u}^{(s)}_h$ on K. In addition, we define

$$\mathbf{z}^{(s)} \;:=\; \mathcal{P}^K_k(\mathbf{u}^{(s)}_{h,K}) \;=\; \sum_{i=1}^{2m} \gamma^K_i \, \boldsymbol{\varphi}^K_i \,.$$

It is important to remark here that, defining $\boldsymbol{\beta}^K := (\beta^K_1, \ldots, \beta^K_{2n^V_k})^{\mathrm{t}}$ and $\boldsymbol{\gamma}^K := (\gamma^K_1, \ldots, \gamma^K_{2m})^{\mathrm{t}}$, there holds

$$\boldsymbol{\gamma}^K \;=\; \mathrm{kron}(\mathbf{I}_2,\, \mathbf{P}_V)\, \boldsymbol{\beta}^K \,,$$

where $\mathbf{P}_V$ is defined in (3.11).

Now, employing the notation $\mathbf{z}^{(s)} := (z^{(s)}_1, z^{(s)}_2)^{\mathrm{t}}$, such that

$$z^{(s)}_\ell \;:=\; \sum_{r=1}^{m} z^{(s)}_{\ell,r} \, \varphi^K_r \,, \qquad \text{for } \ell = 1, 2 \,,$$

we introduce the matrices:

$$\mathbf{M}^{(s)}_{z_\ell}\ell \;:=\; \left[\int_K z^{(s)}_\ell \varphi^K_i \varphi^K_j\right] \;=\; \sum_{r=1}^{m} z^{(s)}_{\ell,r} \left[\int_K \varphi^K_r \varphi^K_i \varphi^K_j\right] \in \mathbb{R}^{m\times m} \,,$$

with $\ell = 1, 2$, whose entries can be calculated by using the formula (3.4). On the other hand, using matrices $\mathbf{M}^{(s)}_{z_\ell}1$ and $\mathbf{M}^{(s)}_{z_\ell}2$, we now introduce the last auxiliary matrices:

$$\mathbf{M}^{(s)}_u \;:=\; \mathrm{kron}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \, \mathbf{M}^{(s)}_{z_\ell} 1\, \mathbf{P}_V\right) + \mathrm{kron}\left(\begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \, \mathbf{M}^{(s)}_{z_\ell} 2\, \mathbf{P}_V\right) \in \mathbb{R}^{(4m)\times(2n^V_k)} \,,$$

and

$$\mathbf{I}_{\mathrm{per}} \;:=\; \mathbf{I}_{4m} \,+\, \mathrm{kron}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \, \mathbf{I}_m\right) .$$

Next, according to the above notation, the operator $\mathbf{G}_{1,K}^{(s)} \in \mathbb{R}^{2n_k^H \times 2n_k^V}$ is given by:

$$\mathbf{G}_{1,K}^{(s)} := \left[\int_K \boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K) : \left[\mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K) \otimes \mathcal{P}_k^K(\mathbf{u}_h^{(s)})\right]^{\mathtt{d}}\right] = (\mathbf{P}^K)^{\mathtt{t}} \left[\int_K (\Phi_i^K)^{\mathtt{d}} : \left[\mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K) \otimes \mathbf{z}^{(s)}\right]\right]$$

$$= (\mathbf{P}^K)^{\mathtt{t}} \left[\begin{array}{c:c} \frac{1}{2}\left[\int_K z_1 \varphi_i^K \, P_k^K(\psi_j^K)\right] & -\frac{1}{2}\left[\int_K z_2 \varphi_i^K \, P_k^K(\psi_j^K)\right] \\ \hdashline \left[\int_K z_2 \varphi_i^K \, P_k^K(\psi_j^K)\right] & \mathbf{0}_{m \times n_k^V} \\ \hdashline \mathbf{0}_{m \times n_k^V} & \left[\int_K z_1 \varphi_i^K \, P_k^K(\psi_j^K)\right] \\ \hdashline -\frac{1}{2}\left[\int_K z_1 \varphi_i^K \, P_k^K(\psi_j^K)\right] & \frac{1}{2}\left[\int_K z_2 \varphi_i^K \, P_k^K(\psi_j^K)\right] \end{array}\right] = (\mathbf{P}^K)^{\mathtt{t}} \, \mathbf{M}_u^{(s)} .$$

Furthermore, in a similar way, we deduce that

$$\mathbf{G}_{2,K}^{(s)} := \left[-\kappa_2 \int_K \boldsymbol{\mathcal{P}}_k^K(\nabla \vec{\boldsymbol{\psi}}_i^K) : \left[\mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K) \otimes \mathcal{P}_k^K(\mathbf{u}_h^{(s)})\right]^{\mathtt{d}}\right] = -\kappa_2 \, (\mathcal{D}\mathbf{P}_V)^{\mathtt{t}} \, \mathbf{M}_u^{(s)} \in \mathbb{R}^{2n_k^V \times 2n_k^V} .$$

Finally, following the previous analysis, it is not difficult to obtain that

$$\begin{aligned} \mathcal{D}\mathbf{G}_{1,K}^{(s)} &:= \left[\int_K \boldsymbol{\mathcal{P}}_k^K(\vec{\boldsymbol{\Psi}}_i^K) : \left[\mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K) \otimes \mathcal{P}_k^K(\mathbf{u}_h^{(s)}) + \mathcal{P}_k^K(\mathbf{u}_h^{(s)}) \otimes \mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K)\right]^{\mathtt{d}}\right] \\ &= (\mathbf{P}^K)^{\mathtt{t}} \, \mathbf{I}_{\mathrm{per}} 0 I \, \mathbf{M}_u^{(s)} \in \mathbb{R}^{2n_k^H \times 2n_k^V} , \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}\mathbf{G}_{2,K}^{(s)} &:= \left[-\kappa_2 \int_K \boldsymbol{\mathcal{P}}_k^K(\nabla \vec{\boldsymbol{\psi}}_i^K) : \left[\mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K) \otimes \mathcal{P}_k^K(\mathbf{u}_h^{(s)}) + \mathcal{P}_k^K(\mathbf{u}_h^{(s)}) \otimes \mathcal{P}_k^K(\vec{\boldsymbol{\psi}}_j^K)\right]^{\mathtt{d}}\right] \\ &= -\kappa_2 \, (\mathcal{D}\mathbf{P}_V)^{\mathtt{t}} \, \mathbf{I}_{\mathrm{per}} 0 I \, \mathbf{M}_u^{(s)} \in \mathbb{R}^{2n_k^V \times 2n_k^V} . \end{aligned}$$

5.5 Calculable approximations of $\boldsymbol{\sigma}$, u, and p

Once Newton's iteration is over, we can use the discrete operators described in previous sections to find nonvirtual approximations of all the unknowns (see [8, Sections 5.3 and 5.4]). Indeed, given $(\boldsymbol{\sigma}_h^{(s)}, \mathbf{u}_h^{(s)}) \in \mathbb{X}_h$ the final approximation of the solution of (5.6), we consider, for each $K \in \mathcal{T}_h$, the vector $\begin{pmatrix} \boldsymbol{\alpha}^K \\ \boldsymbol{\beta}^K \\ \xi_h \end{pmatrix}$ containing the local degrees of freedom of $\boldsymbol{\sigma}_h^{(s)}$, $\mathbf{u}_h^{(s)}$ and $\xi_h^{(s)}$, respectively, sorted as indicated in Fig. 1. More precisely, following (5.11), we have

$$\boldsymbol{\sigma}_h^{(s)}\big|_K = \sum_{i=1}^{2n_k^H} \alpha_i^K \, \vec{\boldsymbol{\Psi}}_i^K \text{ and } \mathbf{u}_h^{(s)}\big|_K = \sum_{i=1}^{2n_k^V} \beta_i^K \, \vec{\boldsymbol{\psi}}_i^K ,$$

where $\boldsymbol{\alpha}^K = (\alpha_1^K, \ldots, \alpha_{2n_k^H}^K)^{\mathtt{t}}$ and $\boldsymbol{\beta}^K = (\beta_1^K, \ldots, \beta_{2n_k^V}^K)^{\mathtt{t}}$. Now, according to [8, eq. (5.32)]), we introduce the fully computable local approximations of $\boldsymbol{\sigma}_h^{(s)}$ and $\mathbf{u}_h^{(s)}$ given by

$$\widehat{\boldsymbol{\sigma}}_h\big|_K \,:=\, \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\sigma}_h^{(s)}\big|_K) \,=\, \sum_{i=1}^{4m} a_i^K\, \Phi_i^K\,,$$

and

$$\widehat{\mathbf{u}}_h\big|_K \,:=\, \mathcal{P}_k^K(\mathbf{u}_h^{(s)}\big|_K) \,=\, \sum_{i=1}^{2m} b_i^K\, \boldsymbol{\varphi}_i^K\,,$$

respectively. Moreover, it is quite simple to see that

$$\boldsymbol{a}^K \,=\, \mathbf{P}^K \boldsymbol{\alpha}^K \text{ and } \boldsymbol{b}^K \,=\, \mathrm{kron}(\mathbf{I}_2,\, \mathbf{R}^K)\, \boldsymbol{\beta}^K\,,$$

where $\boldsymbol{a}^K = (a_1^K, \ldots, a_{4m}^K)^{\mathtt{t}}$ and $\boldsymbol{b}^K = (b_1^K, \ldots, b_{2m}^K)^{\mathtt{t}}$. In addition, we now present the following computable approximation of the pressure:

$$\widehat{p}_h\big|_K \,:=\, -\frac{1}{2}\,\mathrm{tr}\Big(\widehat{\boldsymbol{\sigma}}_h\big|_K \,+\, \widehat{c}_h\,\mathbf{I}_2 \,+\, \widehat{\mathbf{u}}_h\big|_K \otimes \widehat{\mathbf{u}}_h\big|_K\Big)\,,$$

with $\widehat{c}_h \,:=\, -\frac{1}{2|\Omega|}\,\|\widehat{\mathbf{u}}_h\|_{0,\Omega}^2$. Finally, in [8, Section 5.4], a second approximation $\widetilde{\boldsymbol{\sigma}}_h$ of the pseudostress $\boldsymbol{\sigma}$, which yields optimal rate of convergence in the broken $\mathbb{H}(\mathbf{div};\Omega)$-norm is presented. Here, the calculation of $\widetilde{\boldsymbol{\sigma}}_h$ is not presented, since it follows similarly as the previous operators described before.

5.6 Numerical results

In this section, we present a numerical experiment in order to illustrate the performance of the mixed virtual element scheme (5.6) employing Newton's iteration introduced in (5.8). It allows us to validate the operators introduced in Section 4, together with the numerical experiments presented in recent papers about mixed-VEM schemes, which utilized our implementation approach (see [7, 9, 15, 24–26]). We begin by recalling from (5.10) that N stands for the total number of degrees of freedom (unknowns) of (5.8). In addition, the individual errors are defined by

$$\mathsf{e}(\boldsymbol{\sigma}) \,:=\, \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,, \qquad \mathsf{e}(\mathbf{u}) \,:=\, \|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega}\,, \qquad \mathsf{e}(\widehat{\mathbf{u}}) \,:=\, \left\{\sum_{K\in\mathcal{T}_h} \|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{1,K}^2\right\}^{1/2},$$

$$\mathsf{e}(p) \,:=\, \|p - \widehat{p}_h\|_{0,\Omega}\,, \text{ and } \mathsf{e}(\widetilde{\boldsymbol{\sigma}}) \,:=\, \left\{\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{div};K}^2\right\}^{1/2},$$

where $\widehat{\boldsymbol{\sigma}}_h$, $\widehat{\mathbf{u}}_h$, $\widehat{p}_h$, and $\widetilde{\boldsymbol{\sigma}}_h$ are introduced in Section 5.5. In turn, the associated experimental rates of convergence are given by

$$\mathtt{r}(\cdot) \,:=\, \frac{\log\big(\mathsf{e}(\cdot)\,/\,\mathsf{e}'(\cdot)\big)}{\log(h\,/\,h')}\,,$$

where e and e' denote the corresponding errors for two consecutive meshes with sizes h and h', respectively.

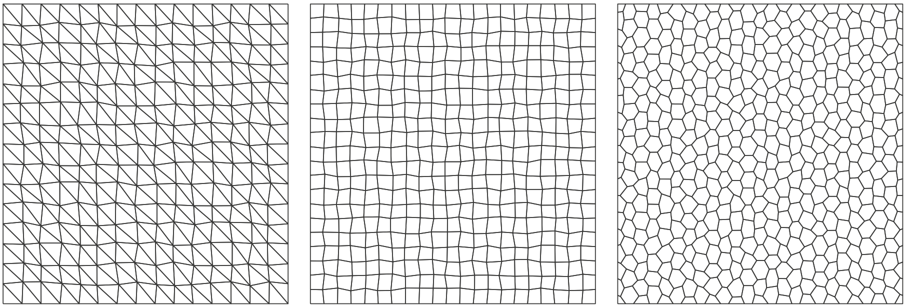


Fig. 2 Examples of the meshes to be used in the calculations

The Newton method (5.8) is solved by using a tolerance of 10^{-6} and taking as initial iteration the solution of the associated linear Stokes problem, where four iterations were required to achieve the given tolerance. On the other hand, the numerical results presented below were obtained using a MATLAB code, where the corresponding linear systems were solved using its instruction “\” as main solver.

Next, we consider $\Omega := (-0.5, 1.5) \times (0, 2)$, $\mu = 0.1$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by the flow from [33], that is,

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} 1 - \exp(\lambda x_1)\cos(2\pi x_2) \\ \dfrac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2) \end{pmatrix} \text{ and } p(\mathbf{x}) = \frac{1}{2}\exp(2\lambda x_1) - \frac{1}{8\lambda}\big\{\exp(3\lambda) - \exp(-\lambda)\big\},$$

for all $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$, where $\lambda := \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}$ and $Re := \mu^{-1} = 10$ is the Reynolds number. Moreover, according to [8, Theorem 2.1], we set the parameters

Table 1 History of convergence using triangles

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\widehat{\mathbf{u}})$	$\mathtt{r}(\widehat{\mathbf{u}})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$
0	0.1230	4419	3.02e+00	−−	4.99e-01	−−	1.43e+01	−−	1.39e+00	−−	6.90e+00	−−
	0.0943	7443	2.24e+00	1.13	3.50e-01	1.34	1.43e+01	0.00	9.76e-01	1.33	5.28e+00	1.00
	0.0488	27379	1.05e+00	1.15	1.44e-01	1.34	1.43e+01	0.00	3.98e-01	1.36	2.70e+00	1.02
	0.0354	51843	7.36e-01	1.10	9.63e-02	1.25	1.43e+01	0.00	2.65e-01	1.26	1.95e+00	1.01
	0.0283	80803	5.79e-01	1.07	7.40e-02	1.18	1.43e+01	0.00	2.03e-01	1.19	1.56e+00	1.01
1	0.1230	19415	2.19e-01	−−	2.29e-02	−−	2.19e+00	−−	9.98e-02	−−	4.55e-01	−−
	0.0943	32883	1.29e-01	1.98	1.29e-02	2.15	1.68e+00	1.00	5.85e-02	2.01	2.69e-01	1.97
	0.0488	122035	3.52e-02	1.98	3.35e-03	2.05	8.71e-01	1.00	1.56e-02	2.00	7.27e-02	1.99
	0.0354	231683	1.86e-02	1.98	1.76e-03	2.01	6.32e-01	1.00	8.23e-03	2.00	3.83e-02	1.99
	0.0283	361603	1.19e-02	1.98	1.12e-03	2.00	5.05e-01	1.00	5.27e-03	2.00	2.45e-02	1.99
2	0.1230	40759	1.85e-02	−−	1.07e-03	−−	1.75e-01	−−	7.84e-03	−−	2.52e-02	−−
	0.0943	69123	8.39e-03	2.98	4.72e-04	3.07	1.03e-01	2.00	3.54e-03	2.99	1.14e-02	2.98
	0.0488	257059	1.16e-03	3.00	6.45e-05	3.02	2.75e-02	2.00	4.86e-04	3.01	1.59e-03	2.99
	0.0354	488323	4.43e-04	3.00	2.46e-05	3.00	1.45e-02	2.00	1.84e-04	3.02	6.07e-04	3.00
	0.0283	762403	2.27e-04	3.00	1.26e-05	3.00	9.27e-03	2.00	9.38e-05	3.02	3.11e-04	3.00

Table 2 History of convergence using quadrilaterals

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\widehat{\mathbf{u}})$	$\mathtt{r}(\widehat{\mathbf{u}})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$
0	0.1008	6403	3.03e+00	−−	5.44e-01	−−	1.43e+01	−−	1.79e+00	−−	6.30e+00	−−
	0.0787	10417	2.20e+00	1.30	3.85e-01	1.39	1.43e+01	0.00	1.25e+00	1.45	4.88e+00	1.03
	0.0404	39043	8.69e-01	1.39	1.49e-01	1.42	1.43e+01	0.00	4.25e-01	1.61	2.40e+00	1.06
	0.0307	66993	6.04e-01	1.34	1.02e-01	1.40	1.43e+01	0.00	2.73e-01	1.63	1.81e+00	1.04
	0.0229	120417	4.16e-01	1.27	6.89e-02	1.32	1.43e+01	0.00	1.71e-01	1.58	1.33e+00	1.03
1	0.1008	23043	1.57e-01	−−	1.81e-02	−−	1.82e+00	−−	5.86e-02	−−	3.16e-01	−−
	0.0787	37641	9.59e-02	2.00	1.10e-02	2.00	1.42e+00	0.99	3.50e-02	2.08	1.95e-01	1.95
	0.0404	142083	2.49e-02	2.02	2.88e-03	2.01	7.29e-01	1.00	8.78e-03	2.07	5.12e-02	2.00
	0.0307	244233	1.44e-02	2.02	1.67e-03	2.00	5.56e-01	1.00	5.04e-03	2.04	2.97e-02	2.01
	0.0229	439641	7.95e-03	2.01	9.28e-04	2.00	4.14e-01	1.00	2.77e-03	2.03	1.65e-02	2.00
2	0.1008	45827	1.34e-02	−−	8.58e-04	−−	1.30e-01	−−	4.55e-03	−−	1.60e-02	−−
	0.0787	74951	6.41e-03	2.98	4.08e-04	3.00	7.96e-02	1.99	2.17e-03	3.00	7.69e-03	2.96
	0.0404	283523	8.55e-04	3.02	5.45e-05	3.01	2.09e-02	2.00	2.81e-04	3.05	1.03e-03	3.01
	0.0307	487623	3.77e-04	3.01	2.41e-05	3.00	1.21e-02	2.00	1.23e-04	3.04	4.54e-04	3.01
	0.0229	878151	1.55e-04	3.00	9.96e-06	3.00	6.74e-03	2.00	5.08e-05	3.00	1.88e-04	3.00

$\kappa_1 = \kappa_2 = \kappa_3 = 0.1$. In addition, we employ polynomial degrees $k \in \{0, 1, 2\}$, and for the decompositions of Ω used in our computations, we consider triangles, distorted squares and distorted hexagons, as illustrated in the Fig. 2.

In Tables 1, 2, and 3, we summarize the convergence history of the mixed virtual element scheme (5.6) as applied to the present example. It follows from [8, Theorem 5.6] that the rate of convergence $O(h^{k+1})$ is attained by $\mathtt{e}(\boldsymbol{\sigma})$, $\mathtt{e}(\mathbf{u})$, $\mathtt{e}(p)$,

Table 3 History of convergence using hexagons

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\widehat{\mathbf{u}})$	$\mathtt{r}(\widehat{\mathbf{u}})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$
0	0.0959	10535	2.42e+00	−−	4.43e-01	−−	1.43e+01	−−	1.27e+00	−−	5.24e+00	−−
	0.0732	17897	1.83e+00	1.05	3.22e-01	1.18	1.43e+01	0.00	8.92e-01	1.31	4.07e+00	0.94
	0.0527	34143	1.16e+00	1.39	1.98e-01	1.48	1.43e+01	0.00	5.15e-01	1.67	2.92e+00	1.01
	0.0390	61887	7.77e-01	1.34	1.32e-01	1.36	1.43e+01	0.00	3.17e-01	1.62	2.17e+00	0.99
	0.0301	103495	5.66e-01	1.22	9.56e-02	1.23	1.43e+01	0.00	2.12e-01	1.54	1.67e+00	1.01
1	0.0959	31707	1.55e-01	−−	1.90e-02	−−	1.87e+00	−−	5.23e-02	−−	2.84e-01	−−
	0.0732	53681	9.10e-02	1.97	1.12e-02	1.97	1.44e+00	0.97	3.02e-02	2.03	1.68e-01	1.94
	0.0527	102623	4.76e-02	1.98	5.84e-03	1.97	1.04e+00	0.99	1.55e-02	2.04	9.00e-02	1.90
	0.0390	185651	2.64e-02	1.97	3.23e-03	1.98	7.72e-01	0.99	8.46e-03	2.01	5.02e-02	1.94
	0.0301	310827	1.56e-02	2.01	1.93e-03	1.99	5.97e-01	0.99	4.99e-03	2.03	2.98e-02	2.01
2	0.0959	59263	1.55e-02	−−	9.46e-04	−−	1.39e-01	−−	5.96e-03	−−	1.78e-02	−−
	0.0732	100199	7.20e-03	2.85	4.30e-04	2.93	8.18e-02	1.95	2.79e-03	2.82	8.22e-03	2.88
	0.0527	191707	2.75e-03	2.93	1.61e-04	2.98	4.25e-02	1.99	1.04e-03	2.99	3.17e-03	2.90
	0.0390	346543	1.14e-03	2.94	6.57e-05	2.99	2.34e-02	1.99	4.36e-04	2.91	1.31e-03	2.94
	0.0301	580463	5.26e-04	2.98	3.03e-05	2.99	1.39e-02	1.99	2.01e-04	2.97	6.02e-04	2.99

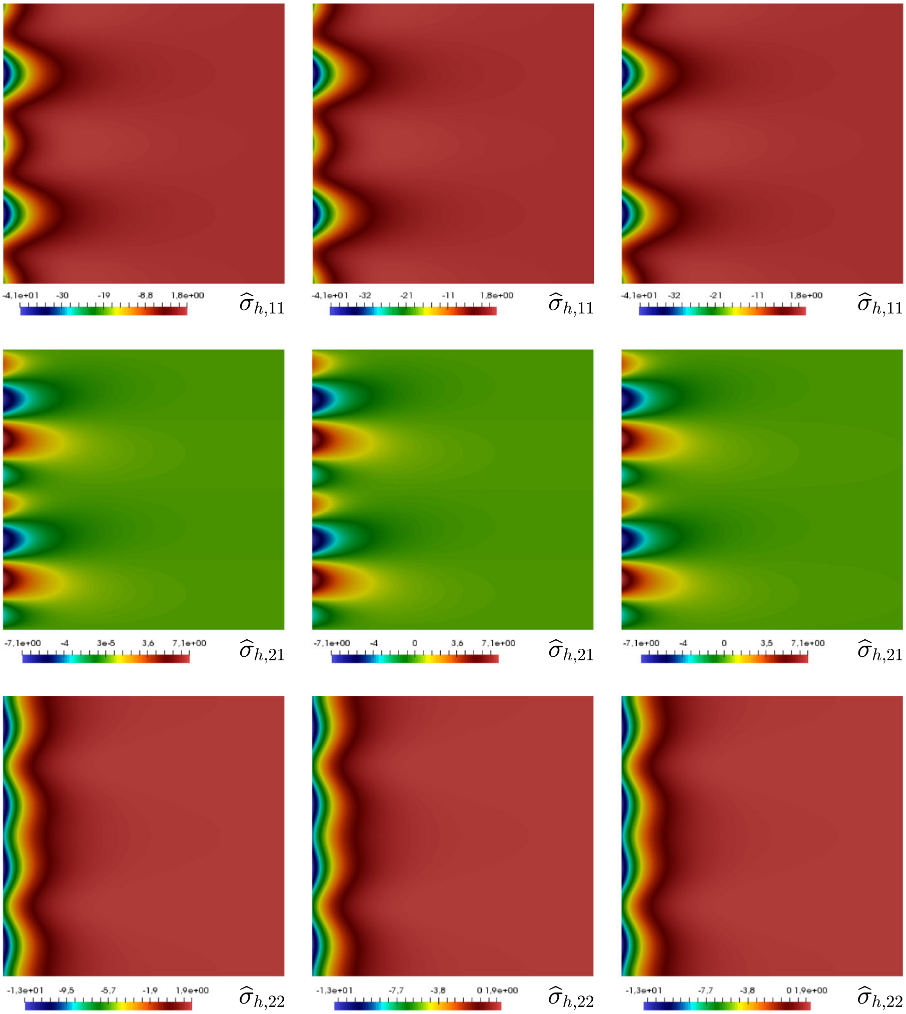


Fig. 3 Example 2, $\widehat{\sigma}_{h,11}$ (top), $\widehat{\sigma}_{h,21}$ (center), and $\widehat{\sigma}_{h,22}$ (bottom), using $k = 2$ and the fifth mesh of triangles (left column), quadrilaterals (center column), and hexagons (right column)

and $\mathsf{e}(\widetilde{\boldsymbol{\sigma}})$, whereas the rate of convergence $O(h^k)$ is attained by $\mathsf{e}(\widehat{\mathbf{u}})$ in this smooth example, for triangular as well as for quadrilateral and hexagonal meshes. Hence, the results validate the suitable behavior of our computational implementation, along with the analysis carried out in [8]. Finally, in Figs. 3 and 4 we display some components of the approximate solutions obtained in this section. They all correspond to those obtained with the last mesh of each kind (triangles, quadrilaterals and hexagons, respectively) and for the polynomial degree $k = 2$.

We end this paper by remarking some possible future directions. It would be interesting to explain some computational aspects about the adaptivity and the a posteriori error estimates for mixed virtual element schemes (see, e.g., [15, 34]). In addition,

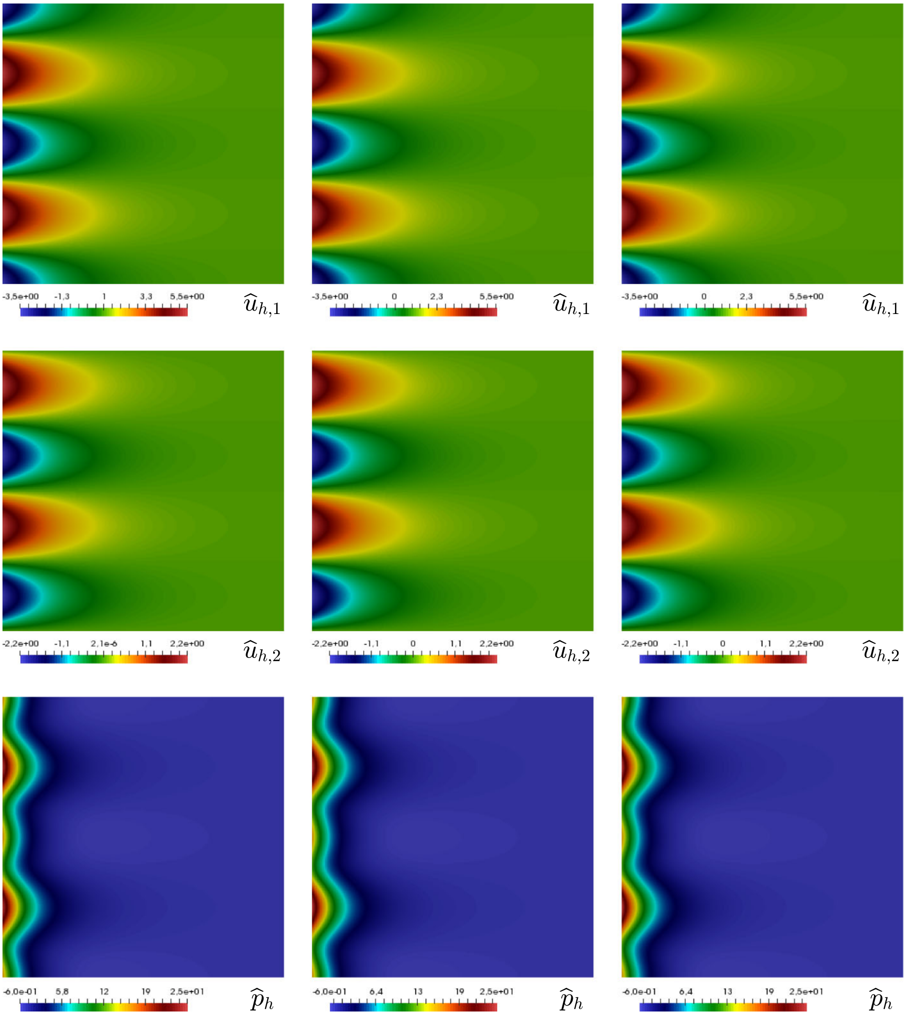


Fig. 4 Example 2, $\widehat{u}_{h,1}$ (top), $\widehat{u}_{h,2}$ (center), and $\widehat{p}_h$ (bottom), using $k = 2$ and the fifth mesh of triangles (left column), quadrilaterals (center column), and hexagons (right column)

in this work, we used direct solvers for solving each lineal system. However, we are interested in developing an algebraic or semi-algebraic multilevel preconditioner in order to employ iterative solvers (see, e.g., [35, 36]).

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