A mixed virtual element method for the Brinkman problem

Ernesto Cáceres∗,† and Gabriel N. Gatica‡

CI²MA, Departamento de Ingeniería Matemática,
Universidad de Concepción, Casilla 160-C, Concepción, Chile
∗ecaceresv@udec.cl; ernesto.caceres.valenzuela@brown.edu
†ggatica@ci2ma.udec.cl

Filander A. Sequeira
Escuela de Matemática, Universidad Nacional,
Campus Omar Dengo, Heredia, Costa Rica
filander.sequeira@una.cr

Received 30 September 2016
Revised 13 January 2017
Accepted 22 January 2017
Published 21 March 2017
Communicated by F. Brezzi

In this paper, we introduce and analyze a mixed virtual element method (mixed-VEM) for the two-dimensional Brinkman model of porous media flow with non-homogeneous Dirichlet boundary conditions. More precisely, we employ a dual-mixed formulation in which the only unknown is given by the pseudostress, whereas the velocity and pressure are computed via postprocessing formulae. We first recall the corresponding variational formulation, and then summarize the main mixed-VEM ingredients that are required for our discrete analysis. In particular, in order to define a calculable discrete bilinear form, whose continuous version involves deviatoric tensors, we propose two well-known alternatives for the local projector onto a suitable polynomial subspace, which allows the explicit integration of these terms. Next, we show that the global discrete bilinear form satisfies the hypotheses required by the Lax–Milgram lemma. In this way, we conclude the well-posedness of our mixed-VEM scheme and derive the associated a priori error estimates for the virtual solution as well as for the fully computable projection of it. Furthermore, we also introduce a second element-by-element postprocessing formula for the pseudostress, which yields an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\text{div})$-norm. Finally, several numerical results illustrating the good performance of the method and confirming the theoretical rates of convergence are presented.

Keywords: Brinkman model; mixed virtual element method; a priori error analysis; postprocessing techniques; high-order approximations.

AMS Subject Classification: 65N30, 65N12, 65N15, 65N99, 76M25, 76S05

∗Corresponding author
†Present Address: Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.
1. Introduction

The Brinkman system, which describes the flow of a viscous fluid in a highly porous medium, and can be considered as a parameter-dependent combination of both the Darcy and Stokes equations, has become a very relevant model of study for numerical analysts of boundary value problems in fluid mechanics during the last few years (see, e.g. Refs. 2–4, 21, 22, 25, 26, 32, and the references therein). In particular, one of the most significant features of this problem arises from its relationship with the evolutive Stokes equations when a time stepping method is applied to them. Having said the above, and since we are mainly interested in mixed variational formulations, we first mention that, up to our knowledge, just a few references have dealt with mixed finite elements or related methods for this problem (see Refs. 21, 27, 28 and 32). In particular, a dual mixed framework and a $H$(div)-conforming finite elements combined with the symmetric interior penalty Galerkin method is employed in Ref. 27 to obtain a stable discrete formulation with respect to a mesh-dependent norm. The corresponding numerical study of the method proposed in Ref. 27 is performed in Ref. 28, in which, besides verifying the theoretical error estimates, the analysis is extended to the case of a non-constant permeability. In turn, a mixed formulation in which the flow vorticity is considered as an additional unknown is introduced in Ref. 32. This approach differs from other vorticity-based formulations since it uses the de Rham sequence to derive an equation for this unknown, instead of obtaining it by taking the curl of the momentum equation.

Nevertheless, we also remark that no stress-based or pseudostress-based methods seemed to be available until the recent contribution. Indeed, a new dual-mixed approach for the two-dimensional Brinkman equations, which includes an alternative way of dealing with mixed boundary conditions and the corresponding a priori and a posteriori error analyses, was introduced and analyzed there. More precisely, the pseudostress $\sigma$ is the main unknown of the resulting saddle point problem in Ref. 21, whereas the velocity and pressure of the fluid are easily recovered in terms of $\sigma$ through simple postprocessing formulae. In addition, as it is usual for this kind of methods, the Dirichlet boundary condition for the velocity becomes natural in this case, and the Neumann boundary condition, being essential, is imposed weakly through the introduction of the trace of the velocity on that boundary as the associated Lagrange multiplier. In this way, the Babuška–Brezzi theory is applied first in Ref. 21 to establish sufficient conditions for the well-posedness of the resulting continuous and discrete formulations. As a consequence, Raviart–Thomas elements of order $k \geq 0$ for the pseudostress, and continuous piecewise polynomials of degree $k + 1$ for the Lagrange multiplier, become a feasible choice of finite element subspaces. Next, a reliable and efficient residual-based a posteriori error estimator is derived there. Suitable auxiliary problems, the continuous inf-sup conditions satisfied by the bilinear forms involved, a discrete Helmholtz decomposition, and the local approximation properties of the Raviart–Thomas and Clément interpolation operators are the main tools for proving the reliability. In turn, Helmholtz’s
Mixed virtual element method for Brinkman decomposition, inverse inequalities, and the classical localization technique based on triangle-bubble and edge-bubble functions are employed to show the efficiency. Lately, a natural extension of the analysis and results from Ref. 21 to a class of Brinkman models whose viscosity depends nonlinearly on the gradient of the velocity, which is a characteristic feature of quasi-Newtonian Stokes flows, was developed in Ref. 22. A reliable and efficient residual-based a posteriori error estimator was also derived in Ref. 22 by following basically the same approach from Ref. 21.

On the other hand, the virtual element method (VEM), introduced in Ref. 6 for the Poisson problem as a model, is one of the high-order discretization schemes that arose as a natural consequence of new developments and interpretations of the mimetic finite difference method (MFDM) (see, e.g., Ref. 14). The main advantages of VEM approaches include an extension of the classical finite element technique to general polygonal and polyhedral meshes, and also as a generalization of the MFDM to arbitrary degrees of accuracy and arbitrary continuity properties. Additionally, as remarked in Ref. 8, other benefits of VEM, when compared with finite volume methods, MMFD, and related techniques, are given by its solid mathematical ground, the simplicity of the respective computational coding, and the quality of the numerical results provided. While most of the projectors employed originally to define the virtual element schemes were ad hoc to the problem under consideration, it is interesting to highlight that the first attempts to derive a systematic use of the simple $L^2$-projection operator were introduced in Ref. 1, and then in Ref. 12 for the case of non-coercive bilinear forms. Furthermore, in the context of purely mixed virtual element techniques, that is based on dual-mixed variational formulations, the method was initially developed in Ref. 15, and more recently extended in Refs. 9, 10, and 18. In particular, edge and face VEM spaces in 2D and 3D, which together with the nodal and volume spaces constitute a discrete complex, were developed in Ref. 9, whereas Ref. 10 generalizes the results of Ref. 9 to the case of variable coefficients. In turn, Ref. 18 provides the first analysis of a virtual element method for a mixed variational formulation of the Stokes problem in which the pseudostress and the velocity are the only unknowns, whereas the pressure is computed via a postprocessing formula (see also Ref. 17 for further details). Therein, a new local projector onto a suitable space of polynomials is presented, which takes into account the main features of the continuous solution and allows the explicit integration of the terms involving the deviatoric tensors. The uniform boundedness of the resulting family of local projectors is established and its approximation properties are also derived. For several other contributions on VEM and mixed-VEM we refer for instance to Refs. 5, 7, 16, 20 and 30.

According to the above discussion, in this present paper we are interested in continuing the research line drawn by Refs. 21 and 18, and aim to develop a mixed-VEM approach for the two-dimensional Brinkman problem with non-homogeneous Dirichlet boundary conditions. To this end, we first proceed as in Ref. 21 and use both the equilibrium equation and the incompressibility condition to eliminate the velocity and pressure, respectively, so that the pseudostress becomes now
the only unknown. Moreover, in order to define an explicitly computable bilinear form, we take advantage of the particular local projection defined in Ref. 18 as well as of the $L^2$-orthogonal projection introduced and analyzed in Ref. 9 (see also Ref. 10). In other words, we propose two mixed virtual element methods depending on two different projectors. The rest of this work is organized as follows. In Sec. 2, we introduce the boundary value problem of interest, and recall its pseudostress-based mixed formulation and the associated well-posedness result. Then, in Sec. 3 we follow Refs. 9 and 10 to introduce the virtual element subspace that will be employed. This includes the basic assumptions on the polygonal mesh, the definition of the local virtual element space, the projections and interpolants to be employed together with their respective approximation properties, and finally the definition itself of the global virtual element subspaces. Next, in Sec. 4, we introduce a fully calculable local discrete bilinear form, which depends on a suitable projection of the local virtual space, establish its boundedness and related properties, and describe two specific choices of that projection. In turn, in Sec. 5 we first set the corresponding mixed virtual element method, and apply the classical Lax–Milgram lemma to deduce its well-posedness. Then, we employ suitable bounds and identities satisfied by the bilinear form and the projectors and interpolators involved, to derive the a priori error estimates and corresponding rates of convergence for the virtual solution as well as for the computable projection of it. In addition, we follow the ideas from Refs. 23 and 24 to construct a second approximation for the pseudostress variable $\sigma$, which yields an optimal rate of convergence in the broken $H(\text{div})$-norm. Moreover, this new postprocessing formula can be used in general for any $H(\text{div})$-conforming VEM scheme. Finally, some numerical examples showing the good performance of the method, confirming the rates of convergence for regular and singular solutions, and illustrating the accurateness obtained with the approximate solutions, are reported in Sec. 6.

Notations

We end the present section by providing some notations to be used along the paper, including those already employed above. Indeed, given a bounded domain $\Omega \subseteq \mathbb{R}^2$ with boundary $\Gamma$, we let $n$ be the outward unit normal vector on $\Gamma$. In addition, standard terminology will be adopted for Lebesgue spaces $L^p(\Omega)$, $p > 1$, and Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $H^{1/2}(\Gamma)$ is the space of traces of functions of $H^1(\Omega)$, and $H^{-1/2} (\Gamma)$ denotes its dual. By $M$ and $\tilde{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space $M$. Then, letting $\text{div}$ (respectively, rot) be the usual divergence operator $\text{div}$ (respectively, rotational operator rot) acting along the rows of a given tensor, we recall that the spaces:

$$H(\text{div}; \Omega) := \{\tau \in L^2(\Omega) : \text{div}(\tau) \in L^2(\Omega)\},$$

$$H(\text{div}; \Omega) := \{\tau \in L^2(\Omega) : \text{div}(\tau) \in L^2(\Omega)\},$$
$H(\text{rot}; \Omega) := \{ \tau \in L^2(\Omega) : \text{rot}(\tau) \in L^2(\Omega) \}$,

and

$H(\text{div}; \Omega) := \{ \tau \in L^2(\Omega) : \text{div}(\tau) \in L^2(\Omega) \}$,

equipped with the usual norms:

$\|\tau\|_{\text{div};\Omega}^2 := \|\tau\|_{0,\Omega}^2 + \|\text{div}(\tau)\|_{0,\Omega}^2, \quad \forall \tau \in H(\text{div}; \Omega),$

$\|\tau\|_{\text{rot};\Omega}^2 := \|\tau\|_{0,\Omega}^2 + \|\text{rot}(\tau)\|_{0,\Omega}^2, \quad \forall \tau \in H(\text{rot}; \Omega),$

and

$\|\tau\|_{\text{rot};\Omega}^2 := \|\tau\|_{0,\Omega}^2 + \|\text{rot}(\tau)\|_{0,\Omega}^2, \quad \forall \tau \in H(\text{rot}; \Omega)$

are Hilbert spaces. Also, given $\tau := (\tau_{ij})$, $\zeta := (\zeta_{ij}) \in \mathbb{R}^{2 \times 2}$, we write as usual

$\tau^T := (\tau_{ji})$, \quad $\text{tr}(\tau) := \sum_{i=1}^{2} \tau_{ii}$, \quad $\tau^d := \tau - \frac{1}{2} \text{tr}(\tau) I$, \quad and \quad $\tau : \zeta := \sum_{i,j=1}^{2} \tau_{ij} \zeta_{ij},$

where $I$ is the identity matrix of $\mathbb{R}^{2 \times 2}$. Finally, in what follows we employ $0$ to denote a generic null vector, null tensor or null operator, and use $C$, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The Brinkman Problem and Its Pseudostress-Based Formulation

Let $\Omega$ be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma$. Our goal is to determine the velocity $u$, the pseudostress $\sigma$, and the pressure $p$ of a steady Brinkman flow occupying the region $\Omega$. In other words, given a volume force $f \in L^2(\Omega)$ and a Dirichlet datum $g \in H^{1/2}(\Gamma)$, we seek a tensor field $\sigma$, a vector field $u$ and a scalar field $p$ such that:

$\sigma = \mu \nabla u - p I$ in $\Omega$, \quad $\alpha u - \text{div}(\sigma) = f$ in $\Omega,$

$\text{div}(u) = 0$ in $\Omega$, \quad $u = g$ on $\Gamma$, \quad $\int_{\Omega} p = 0,$ \quad (2.1)

where $\mu$ is the dynamic viscosity, and $\alpha > 0$ is a constant approximation of the viscosity divided by the permeability. In addition, as required by the incompressibility condition, we assume from now on that the datum $g$ satisfies the compatibility condition $\int_{\Gamma} g \cdot n = 0$. Furthermore, the incompressible condition also implies that (2.1) can be rewritten as:

$\frac{1}{\mu} \sigma^d = \nabla u$ in $\Omega$, \quad $\alpha u - \text{div}(\sigma) = f$ in $\Omega,$ \quad $u = g$ on $\Gamma$, \quad $\int_{\Omega} \text{tr}(\sigma) = 0,$ \quad (2.2)
where the pressure $p$ can be obtained by the postprocessing formula

$$p = -\frac{1}{2}\text{tr}(\sigma) \quad \text{in } \Omega.$$  

(2.3)

Then, proceeding as in Ref. 21, the velocity is replaced from the second equation of (2.2), that is

$$u = \frac{1}{\alpha}\{f + \text{div}(\sigma)\} \quad \text{in } \Omega,$$  

(2.4)

which yields the following dual-mixed variational formulation of (2.2): Find $\sigma \in H$ such that

$$a(\sigma, \tau) = F(\tau), \quad \forall \tau \in H,$$  

(2.5)

where $H := H_0(\text{div}; \Omega) := \{\tau \in H(\text{div}; \Omega) : \int_{\Omega} \text{tr}(\tau) = 0\}$.

The unique solvability of (2.5) is established next.

Lemma 2.1. There exists $c_\Omega > 0$, depending only on $\Omega$, such that

$$c_\Omega \|\tau\|_{0, \Omega}^2 \leq \|\text{tr}(\tau)\|^2_{0, \Omega} + \|\text{div}(\tau)\|^2_{0, \Omega}, \quad \forall \tau \in H.$$

Proof. See Proposition 9.1.1 of Ref. 13.

Then the $H$-ellipticity of $a$ is proved as follows.

Lemma 2.2. There exists $\eta > 0$, depending only on $\mu$, $\alpha$ and $\Omega$, such that

$$a(\zeta, \zeta) \geq \eta \|\zeta\|^2_{\text{div}; \Omega}, \quad \forall \zeta \in H.$$  

Proof. According to the definition of $a$ and Lemma 2.1, we find that for each $\zeta \in H$ there holds

$$a(\zeta, \zeta) = \frac{1}{\mu}\|\zeta\|^2_{0, \Omega} + \frac{1}{\alpha}\|\text{div}(\zeta)\|^2_{0, \Omega} \geq \eta \|\zeta\|^2_{\text{div}; \Omega},$$

where $\eta := \min\{c_\Omega \eta_0, \frac{1}{2\alpha}\}$ and $\eta_0 := \min\{\frac{1}{\mu}, \frac{1}{2\alpha}\}.$

The unique solvability of (2.5) is established next.
Theorem 2.1. Assume that $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Then, there exists a unique solution $\sigma \in H$ to (2.5). In addition, there exists $C > 0$ such that

$$\|\sigma\|_{\text{div};\Omega} \leq C\{\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}\}.$$ 

Proof. Thanks to the ellipticity of $a$ and the boundedness of $F$, the proof is a straightforward application of the Lax–Milgram lemma. \hfill \Box

We end this section by remarking that the extreme situations given by the Stokes and diffusion equations, which arise when $\alpha = 0$ and $\mu = 0$, respectively, are not included in the present mixed formulation since in both cases the bilinear form $a$ (cf. (2.6)) becomes undefined.

3. The Virtual Element Subspace

The main purpose of this section is to introduce a virtual element subspace $\mathbb{H}_h$ of $H := H^0(\text{div};\Omega)$, with which we prove later on that the mixed virtual element scheme associated with the continuous formulation (2.5) is well-posed. To this end, we follow the approach from Refs. 9 and 10 to define first a virtual element subspace of $H(\text{div};\Omega)$, and then proceed row-wise to extend the above to $H(\text{div};\Omega)$. Along the way, local virtual element spaces, suitable associated interpolation operators, and their main approximation properties are also provided. While all these results are available in Refs. 9 and 10, for convenience of the reader we recall here most of the corresponding details.

3.1. Basic assumptions

We begin by letting $\{T_h\}_{h>0}$ be a family of decompositions of $\Omega$ in polygonal elements. Then, for each $K \in T_h$ we let $d_K$ be the number of its edges and denote its diameter by $h_K$. In addition, we define as usual $h := \max\{h_K : K \in T_h\}$. Furthermore, in what follows we assume that there exists a constant $C_T > 0$ such that for each decomposition $T_h$ and for each $K \in T_h$ there hold:

(a) the ratio between the shortest edge and the diameter $h_K$ of $K$ is bigger than $C_T$, and
(b) $K$ is star-shaped with respect to a ball $B$ of radius $C_T h_K$ and center $x_B \in K$, that is, for each $x_0 \in B$, all the line segments joining $x_0$ with any $x \in K$ are contained in $K$, or, equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in $K$.

As a consequence of the above hypotheses, one can show that each $K \in T_h$ is simply connected, and that there exists an integer $N_T$ (depending only on $C_T$), such that for each $K \in T_h$, $d_K$ is bounded above by $N_T$. 
3.2. The local virtual element space

In what follows, given an integer \( \ell \geq 0 \) and \( U \subseteq \mathbb{R}^2 \), we let \( P_{\ell}(U) \) be the space of polynomials defined in \( U \) of total degree at most \( \ell \). Then, for each integer \( k \geq 0 \) and for each \( K \in \mathcal{T}_h \), we introduce the local virtual element space of order \( k \) (see, e.g. Refs. 9 and 10):

\[
H_k^K := \{ \tau := (\tau_1, \tau_2)^T \in H(\text{div}; K) \cap H(\text{rot}; K) : \tau \cdot n|_e \in P_k(e) \forall \text{ edge } e \in \partial K, \quad \text{div}(\tau) \in P_{k+1}(K), \quad \text{rot}(\tau) \in P_{k-1}(K) \},
\]

where \( \text{rot}(\tau) := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2} \) and \( P_{-1}(K) := \{ 0 \} \). In addition, given an edge \( e \in \mathcal{T}_h \) with medium point \( x_e \) and length \( h_e \), we consider the following set of \( k + 1 \) normalized monomials on \( e \):

\[
B_k(e) := \left\{ \left( \frac{x - x_e}{h_e} \right)^j \right\}_{0 \leq j \leq k}, \tag{3.2}
\]

which certainly constitutes a basis of \( P_k(e) \). Similarly, given an integer \( \ell \geq 0 \) and an element \( K \in \mathcal{T}_h \) with barycenter \( x_K \), we define the following set of \( \frac{1}{2}(\ell + 1)(\ell + 2) \) normalized monomials

\[
B_{\ell}(K) := \left\{ \left( \frac{x - x_K}{h_K} \right)^{\alpha} \right\}_{0 \leq |\alpha| \leq \ell}, \tag{3.3}
\]

which is a basis of \( P_{\ell}(K) \). We remark here that (3.3) makes use of the multi-index notation, where, given \( x := (x_1, x_2)^T \in \mathbb{R}^2 \) and \( \alpha := (\alpha_1, \alpha_2)^T \), with non-negative integers \( \alpha_1, \alpha_2 \), we set \( x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \) and \( |\alpha| := \alpha_1 + \alpha_2 \). Next, we recall from Refs. 9 and 10 the following local degrees of freedom for a given \( \tau \in H_k^K \):

\[
m^{n,K}_{q,e}(\tau) := \int_e \tau \cdot n q \quad \forall q \in B_k(e), \quad \forall \text{ edge } e \in \partial K, \\
m^{\text{div},K}_{q,K}(\tau) := \int_K \tau \cdot \nabla q \quad \forall q \in B_{\ell}(K) \setminus \{1\}, \tag{3.4} \\
m^{\text{rot},K}_{q,K}(\tau) := \int_K \tau \cdot q \quad \forall q \in \mathcal{G}_k^K(1),
\]

where \( \mathcal{G}_k^K(1) \) is a basis of \( (\nabla P_{k+1}(K))^\perp \cap P_{k}(K) \), which is the \( L^2(K) \)-orthogonal of \( \nabla P_{k+1}(K) \) in \( P_k(K) \). Then, thanks to the cardinalities of \( B_{\ell}(K) \) and \( B_K(K) \), and according to the dimensions of \( P_k(K) \) and \( \nabla P_{k+1}(K) \) (see also Ref. 17 for details on the latter), we find that the cardinality of \( \mathcal{G}_k^K(1) \) is \( \frac{k}{2} (k+1) \), and hence the amount of local degrees of freedom defined in (3.4) is given by

\[
n_k^K := (k+1)d_K + \left\{ \frac{(k+1)(k+2)}{2} - 1 \right\} + \frac{k(k+1)}{2} = (k+1)(d_K + k + 1) - 1.
\]

Moreover, it is not difficult to prove that for each \( K \in \mathcal{T}_h \) these \( n_k^K \) local degrees of freedom are unisolvent in \( H_k^K \) (see Ref. 9).
According to the above discussion, we are now able to define for each $K \in \mathcal{T}_h$ the tensorial local virtual element space

$$\mathbb{H}_K:=\{\tau \in \mathbb{H}(\text{div}; K) \cap \mathbb{H}(\text{rot}; K): (\tau_{11}, \tau_{12})^T \in \mathbb{H}_K^\tau \quad \forall i \in \{1, 2\}\},$$

which is certainly unisolvent with respect to the $2n_K^\tau$ degrees of freedom:

\begin{align}
\mathbf{m}^\text{div}_{e,K}(\tau) &:= \int_K \tau \cdot \mathbf{q} \quad \forall \mathbf{q} \in \mathcal{B}_h(e), \quad \forall \text{edge } e \in \partial K, \\
\mathbf{m}^\text{rot}_{e,K}(\tau) &:= \int_K \tau : \mathbf{q} \quad \forall \mathbf{q} \in \mathcal{B}_h(K) \setminus \{(1, 0)^T, (0, 1)^T\}, \\
\mathbf{m}^\text{rot}_{\rho,K}(\tau) &:= \int_K \tau : \mathbf{q} \quad \forall \mathbf{q} \in \mathcal{G}_h^r(K),
\end{align}

where

$$\mathcal{B}_h(e) := \{(q, 0)^T : q \in \mathcal{B}_h(e)\} \cup \{(0, q)^T : q \in \mathcal{B}_h(e)\},$$

$$\mathcal{B}_h(K) := \{(q, 0)^T : q \in \mathcal{B}_h(K)\} \cup \{(0, q)^T : q \in \mathcal{B}_h(K)\},$$

and

$$\mathcal{G}_h^r(K) := \left\{\begin{pmatrix} q \\ 0 \end{pmatrix} : q \in \mathcal{G}_h^r(K)\right\} \cup \left\{\begin{pmatrix} 0 \\ q \end{pmatrix} : q \in \mathcal{G}_h^r(K)\right\}.$$
degrees of freedom given by (3.4) do allow the explicit calculation of
respectively. In addition, it is well-known (see, e.g. Ref. 10 or Ref. 18) that, given 
3.3.

\begin{equation}
\mathbf{P}_k(K) = \nabla \mathbf{P}_{k+1}(K) \oplus \mathbf{P}_k(K).
\end{equation}

In this way, and not having in this case the need of applying QR decompositions,
the computational implementation of those degrees of freedom will certainly be
simpler. For further details we refer to [11, Sec. 2.1], and particularly to Eq. (2.10)
of Ref. 11.

3.3. \textit{L}^2\text{-orthogonal projections, interpolants, and}
\textit{approximation properties}

We now let \(P^K_k : \mathbf{L}^2(K) \to \mathbf{P}_k(K)\) and \(\mathcal{P}^K_k : \mathbf{L}^2(K) \to \mathbf{P}_k(K)\) be the orthogonal
projector and its corresponding vectorial version, which, given \(v \in \mathbf{L}^2(K)\) and
\(\mathbf{v} \in \mathbf{L}^2(K)\), are characterized by

\begin{equation}
P^K_k(v) \in \mathbf{P}_k(K) \quad \text{and} \quad \int_K P^K_k(v) q = \int_K v q, \quad \forall q \in \mathbf{P}_k(K)
\end{equation}

and

\begin{equation}
\mathcal{P}^K_k(v) \in \mathbf{P}_k(K) \quad \text{and} \quad \int_K \mathcal{P}^K_k(v) \cdot q = \int_K \mathbf{v} \cdot q, \quad \forall q \in \mathbf{P}_k(K),
\end{equation}

respectively. In addition, it is well-known (see, e.g. Ref. 10 or Ref. 18) that, given
an integer \(m \in \{0, 1, \ldots, k+1\}\), there holds the following approximation properties:

\begin{equation}
||v - P^K_k(v)||_{0,K} \leq Ch^m_K |v|_{m,K} \quad \forall v \in H^m(K), \quad \forall K \in \mathcal{T}_h,
\end{equation}

and

\begin{equation}
||\mathbf{v} - \mathcal{P}^K_k(\mathbf{v})||_{0,K} \leq Ch^m_K |\mathbf{v}|_{m,K}, \quad \forall \mathbf{v} \in H^m(K), \quad \forall K \in \mathcal{T}_h.
\end{equation}

Actually, (3.11) is a direct consequence of (3.10) since it is easy to see that \(\mathcal{P}^K_k(\mathbf{v}) = (P^K_k(v_1), P^K_k(v_2))^T\) for all \(\mathbf{v} := (v_1, v_2)^T \in H^m(K)\).

At this point we observe for later use, as it was remarked in Ref. 10, that the
degrees of freedom given by (3.4) do allow the explicit calculation of \(\mathcal{P}^K_1(\tau)\) for each
\(\tau \in H^1(K)\). Indeed, it suffices to check that the right-hand side of (3.9) is calculable in
this case. To do that, we first note, thanks to the definitions of \(m^a_{q,e}(\tau)\) and \(m^a_{q,K}(\tau)\)
(cf. (3.4)), that we can compute the value of \(\text{div}(\tau) \in \mathbf{P}_k(K)\) by using the identity

\begin{equation}
\int_K \text{div}(\tau) q = -\int_K \tau \cdot \nabla q + \int_{\partial K} \tau \cdot \mathbf{n} q, \quad \forall q \in \mathbf{P}_k(K).
\end{equation}
Next, given $q \in P_h(K)$, we know that there exist unique \( q^\perp \in (\nabla P_{k+1}(K))^\perp \cap P_k(K) \) and \( \tilde{q} \in P_{k+1}(K) \), such that $q = q^\perp + \nabla \tilde{q}$. In this way, it follows that
\[
\int_K \tau \cdot q = \int_K \tau \cdot q^\perp + \int_K \tau \cdot \nabla \tilde{q} = \int_K \tau \cdot q^\perp - \int_K \tilde{q} \text{div}(\tau) + \int_{\partial K} \tau \cdot n \tilde{q},
\]
which, according to (3.12) and the definition of $m^\text{rot}_{q,K}(\tau)$ (cf. (3.4)), yield the required calculation.

Furthermore, we now let $\Pi^k_h : H^1(K) \to H^k_h$ be the interpolation operator with respect to the degrees of freedom (3.4), that is, given $\tau \in H^1(K)$, $\Pi^k_h(\tau)$ is the unique element in $H^k_h$ such that:
\[
\begin{align*}
&m^h_{\text{edge}}(\Pi^k_h(\tau)) := \int_e \tau \cdot n q \quad \forall q \in B_k(e), \quad \forall \text{ edge } e \in \partial K, \\
&m^\text{div}_{q,K}(\Pi^k_h(\tau)) := \int_K \tau \cdot \nabla q \quad \forall q \in B_k(K) \setminus \{1\}, \\
&m^\text{rot}_{q,K}(\Pi^k_h(\tau)) := \int_K \tau \cdot q \quad \forall q \in G^+_k(K).
\end{align*}
\]
Concerning the approximation properties of $\Pi^k_h$, we first recall from Ref. 10 that for each $\tau \in H^m(K)$, with $1 \leq m \leq k + 1$, there holds
\begin{equation}
\|\tau - \Pi^k_h(\tau)\|_{0,K} \leq Ch^m_{K}[\tau|_{m,K}], \quad \forall K \in T_h. \tag{3.13}
\end{equation}

In addition, for each $q \in P_h(K)$ we find that
\[
\int_K \text{div}(\tau - \Pi^k_h(\tau))q = -\int_K (\tau - \Pi^k_h(\tau)) \cdot \nabla q + \int_{\partial K} (\tau - \Pi^k_h(\tau)) \cdot n q = 0,
\]
which, thanks to the fact that $\text{div}(\Pi^k_h(\tau)) \in P_h(K)$, implies that
\begin{equation}
\text{div}(\Pi^k_h(\tau)) = P^k_h(\text{div}(\tau)). \tag{3.14}
\end{equation}

In this way, applying (3.10) we deduce that for each $\tau \in H^1(K)$, such that $\text{div}(\tau) \in H^m(K)$, with $0 \leq m \leq k + 1$, there holds
\begin{equation}
\|\text{div}(\tau - \Pi^k_h(\tau))\|_{0,K} \leq Ch^m_{K}[\text{div}(\tau)|_{m,K}], \quad \forall K \in T_h, \tag{3.15}
\end{equation}
which, together with (3.13), show that for each $\tau \in H^m(K)$ such that $\text{div}(\tau) \in H^m(K)$, with $1 \leq m \leq k + 1$, there holds
\begin{equation}
\|\tau - \Pi^k_h(\tau)\|_{\text{div},K} \leq Ch^m_{K}[|\tau|_{m,K} + |\text{div}(\tau)|_{m,K}], \quad \forall K \in T_h. \tag{3.16}
\end{equation}

Analogously, we now let $\Pi^k_h : \mathbb{H}^1(K) \to \mathbb{H}^k_h$ (cf. (3.5)) be the interpolation operator with respect to the degrees of freedom (3.6). Then, it is straightforward to see that $\Pi^k_h$ reduces to $\Pi^k_h$ acting along each row of a tensor $\tau \in \mathbb{H}^1(K)$, and hence, thanks to (3.16), we conclude now that for each $\tau \in \mathbb{H}^m(K)$ such that $\text{div}(\tau) \in H^m(K)$, with $1 \leq m \leq k + 1$, there holds
\begin{equation}
\|\tau - \Pi^k_h(\tau)\|_{\text{div},K} \leq Ch^m_{K}[|\tau|_{m,K} + |\text{div}(\tau)|_{m,K}], \quad \forall K \in T_h. \tag{3.17}
\end{equation}
3.4. The conforming global virtual element subspaces

We now set the global virtual element subspaces of $H(\text{div}; \Omega)$ and $H(\text{div}; \Omega)$, respectively, that is

$$H^h_k := \{ \tau \in H(\text{div}; \Omega) : \tau|_K \in H^K_k, \ \forall \ K \in T_h \}$$

and

$$H^h_k := \{ \tau \in H(\text{div}; \Omega) : \tau|_K \in H^K_k, \ \forall \ K \in T_h \},$$

or, equivalently

$$H^h_k := \{ \tau \in H(\text{div}; \Omega) : (\tau_{i1}, \tau_{i2})^t \in H^h_k, \ \forall \ i \in \{1, 2\} \}. \quad (3.19)$$

Then, we set $H^h_0(\Omega) := \{ \tau \in H(\text{div}; \Omega) : \tau|_K \in H^1(K) \ \forall \ K \in T_h \}$ and introduce the global interpolation operators $\Pi^h_k : H^h_0(\Omega) \to H^h_k$ and $\Pi^h_k : H^h_0(\Omega) \to H^h_k$, whose local restrictions are given for each $K \in T_h$ by

$$\Pi^h_0(\tau)|_K := \Pi^K_0(\tau|_K) \ \forall \ \tau \in H^h_0(\Omega) \ \text{and} \ \Pi^h_0(\tau)|_K := \Pi^K_0(\tau|_K) \ \forall \ \tau \in H^h_0(\Omega).$$

Note that the well-definiteness of $\Pi^h_0$ (respectively $\Pi^h_0$) is guaranteed by the unisolvency in $H^K_0$ (respectively $H^h_0$) of the local degrees of freedom (3.4) (respectively (3.6)), and by the fact that the belonging $\Pi^h_0(\tau) \in H(\text{div}; \Omega)$ (respectively $\Pi^h_0(\tau)$ in $H(\text{div}; \Omega)$) follows from the definition of the degrees of freedom $m^h_0$ (respectively, $m^h_0$). Furthermore, we remark that the approximation properties of $\Pi^h_0$ and $\Pi^h_0$ follow straightforwardly from those of their local restrictions, which are given by (3.16) and (3.17), respectively.

4. The Local Discrete Bilinear Form

The purpose of this section is to define a computable discrete version $a^K_h : H^h_0(\Omega) \times H^h_0(\Omega) \to \mathbb{R}$ of the local bilinear form $a^K : H(\text{div}; K) \times H(\text{div}; K) \to \mathbb{R}$, which is given for each $K \in T_h$ by

$$a^K(\zeta, \tau) := \frac{1}{\mu} \int_K \zeta^d : \tau^d + \frac{1}{\alpha} \int_K \text{div} (\zeta) \cdot \text{div} (\tau), \ \forall \ \zeta, \tau \in H(\text{div}; K). \quad (4.1)$$

Indeed, the aforementioned goal is motivated by the fact that $a^K$ is not explicitly calculable for $\zeta, \tau \in H^h_0(\Omega)$ since in general the deviatoric tensors $\zeta^d$ and $\tau^d$ are not known on each $K \in T_h$. In order to overcome this difficulty, we first assume the existence of an abstract space $\tilde{H}^h_0(K)$ and a corresponding projector $\bar{\Pi}^h_0 : H(\text{div}; K) \to \tilde{H}^h_0(K)$, both satisfying suitable conditions, so that $a^K_h$ is expressed later on in terms of these projections instead of the original elements of $H^h_0$. Next, a special bilinear form depending on the degrees of freedom defining $H^h_0(K)$ and $\tilde{H}^h_0(K)$, and which is also utilized to define $a^K_h$, is introduced. Then, the explicit definition of $a^K_h$ is provided, and its main boundedness and positivity properties are established. Finally, specific examples of $\tilde{H}^h_0(K)$ and $\bar{\Pi}^h_0$ verifying the indicated conditions are described.
4.1. A suitable projection of the local virtual space

In what follows we assume that for each $K \in \mathcal{T}_h$ there exist a subspace $\bar{\mathcal{H}}^K$ of $\mathbb{H}(\text{div}; K)$ and a projection $\Pi^K_h : \mathbb{H}(\text{div}; K) \to \bar{\mathcal{H}}^K$ satisfying the following abstract assumptions:

(A.1) $\bar{\mathcal{H}}^K \subseteq \mathcal{P}_t(K)$ for some integer $t \geq 0$,
(A.2) $\Pi^K_h(\zeta)$ is explicitly calculable $\forall \zeta \in \mathbb{H}^K_h$,
(A.3) there exists $\hat{C} > 0$, independent of $K$, such that
\[
\|\Pi^K_h(\zeta)\|_{0, K} \leq \hat{C}\|\zeta\|_{\text{div}, K}, \quad \forall \zeta \in \mathbb{H}(\text{div}; K),
\]
(A.4) $\int_K (\Pi^K_h(\zeta))^d : (\Pi^K_h(\tau))^d = \int_K (\bar{\Pi}^K_h(\zeta))^d : \tau^d$, $\forall \zeta, \tau \in \mathbb{H}(\text{div}; K)$, and
(A.5) given an integer $1 \leq m \leq k + 1$, there exists a constant $C > 0$, independent of $K$, such that
\[
\|\zeta - \bar{\Pi}^K_h(\zeta)\|_{0, K} \leq Ch^K_m \|\zeta\|_{m, K},
\]
for all $\zeta \in \mathbb{H}^m(K)$, or at least for all $\zeta \in \mathbb{H}^m_\text{curl}(K)$, where
\[
\mathbb{H}^m_\text{curl}(K) := \{\zeta \in \mathbb{H}^m(K) : \zeta^d = \nabla \text{curl}(w) \text{ for some } w \in \mathbb{H}^{m+2}(K)\}.
\]

According to these assumptions, we first observe that (A.1) and (A.2) guarantee that for each $\zeta \in \mathbb{H}^K_h$, $\bar{\Pi}^K_h(\zeta)$ is known in the whole $K$. In addition, thanks to (A.3) and (A.4) we can carry out the a priori error analysis (see Sec. 5.1 below) of our mixed virtual element scheme (5.3). Finally, we observe in advance that the assumption (A.5) is required to prove later on the optimal rates of convergence for the pseudostress variable $\sigma$. To this regard, note from the first equation in (2.2) and the third equation in (2.1) that $\sigma^d = \mu\nabla u \in \mathbb{L}^2(\Omega)$ and $\text{div}(u) = 0$, respectively. Thus, there exists $w \in \mathbb{H}^2(\Omega)$ such that $u = \text{curl}(w) := (\partial_{x_2} w_1 - \partial_{x_1} w_5)^5$, and hence $\sigma^d = \mu\nabla \text{curl}(w)$. These remarks have motivated, similarly as done in Ref. 18, the introduction of the space $\mathbb{H}^m_\text{curl}(K)$.

4.2. The degrees of freedom-based bilinear form

We now consider $K \in \mathcal{T}_h$ and gather all the $K$-moments of a given $\tau \in \mathbb{H}^1(K)$ (cf. (3.4)) in the set $\{m_{i,K}(\tau)\}_{i=1}^{n^K}$. Then, as usual we let $\{\phi_{j,K}\}_{j=1}^{n^K}$ be the canonical basis of $\mathbb{H}^K_h$, that is, given $i \in \{1, 2, \ldots, n^K\}$, $\phi_{i,K}$ is the unique element in $\mathbb{H}^K_h$ such that
\[
m_{j,K}(\phi_{i,K}) = \delta_{ij}, \quad \forall j \in \{1, 2, \ldots, n^K\}.
\]
It follows easily that
\[
\Pi^K_h(\tau) := \sum_{j=1}^{n^K} m_{j,K}(\tau)\phi_{j,K}.
\]
In particular, it is also clear that

\[ m_{j,K}(\Pi^K_h(\tau)) = m_{j,K}(\tau), \quad \forall j \in \{1, 2, \ldots, n^K_h\}, \quad \forall \tau \in H^1(K). \]

In particular, it is also clear that

\[ \tau := \sum_{j=1}^{n^K_h} m_{j,K}(\tau) \varphi_{j,K}, \quad \forall \tau \in H^K_h. \]

Now, for each \( K \in T_h \) we let \( s^K : H^K_h \times H^K_h \to \mathbb{R} \) be the bilinear form associated with the identity matrix in \( \mathbb{R}^{n^K_h \times n^K_h} \) with respect to the basis \( \{ \varphi_{j,K} \}_{j=1}^{n^K_h} \) of \( H^K_h \), that is

\[ s^K(\tau, \zeta) := \sum_{i=1}^{n^K_h} m_{i,K}(\tau) m_{i,K}(\zeta), \quad \forall \tau, \zeta \in H^K_h. \quad (4.3) \]

In this regard we recall from Eqs. (45) and (104) of Ref. 12 (see also Eq. (5.8) of Ref. 15) that there exist \( c_0, c_1 > 0 \), depending only on \( C_T \), such that

\[ c_0 \| \tau \|^2_{0,K} \leq s^K(\tau, \tau) \leq c_1 \| \tau \|^2_{0,K}, \quad \forall \tau \in H^K_h, \quad \forall K \in T_h. \quad (4.4) \]

Then, we let \( S^K : H^K_h \times H^K_h \to \mathbb{R} \) be the bilinear form associated with the degrees of freedom of \( H^K_h \) (cf. (3.6)), that is

\[ S^K(\tau, \zeta) := \sum_{i=1}^{2} \sum_{j=1}^{2} m^K_h(\tau_{ij}, \zeta_{ij}), \quad \forall \tau := (\tau_{ij}), \quad \zeta := (\zeta_{ij}) \in H^K_h, \quad (4.5) \]

which, due to (4.4), satisfies

\[ c_0 \| \tau \|^2_{0,K} \leq S^K(\tau, \tau) \leq c_1 \| \tau \|^2_{0,K}, \quad \forall \tau \in H^K_h, \quad \forall K \in T_h. \quad (4.6) \]

### 4.3. The computable local discrete bilinear form

Having provided the above analysis, we now let \( a^K_h : H^K_h \times H^K_h \to \mathbb{R} \) be the local discrete bilinear form given by

\[
\begin{align*}
a^K_h(\zeta, \tau) := & \frac{1}{\mu} \int_K \hat{\Pi}^K_h(\zeta) : (\hat{\Pi}^K_h(\tau))^T + \frac{1}{\alpha} \int_K \text{div}(\zeta) \cdot \text{div}(\tau) \\
& + S^K(\zeta - \hat{\Pi}^K_h(\zeta), \tau - \hat{\Pi}^K_h(\tau)), \quad \forall \zeta, \tau \in \hat{H}^K_h.
\end{align*}
\]

Then, we have the following result, which is consequence of the properties of \( \hat{\Pi}^K_h \) and (4.6).

**Lemma 4.1.** For each \( K \in T_h \), there holds

\[ a^K_h(\zeta, \tau) = a^K(\zeta, \tau), \quad \forall \zeta \in \hat{H}^K_h, \quad \forall \tau \in H^K_h, \quad (4.8) \]

and there exist constants \( \alpha_1, \alpha_2 > 0 \), independent of \( h \) and \( K \), such that for all \( \zeta, \tau \in \hat{H}^K_h \) there hold

\[ |a^K_h(\zeta, \tau)| \leq \alpha_1 \| \zeta \|_{\text{div;}K} \| \tau \|_{\text{div;}K}, \quad (4.9) \]
and
\[
\alpha_2 \{ \| \zeta^d \|_{0,K}^2 + \| \text{div} (\zeta) \|_{0,K}^2 \} \leq a_h^K (\zeta, \zeta) \leq \alpha_1 \| \zeta \|_{\text{div},K}^2. \tag{4.10}
\]

**Proof.** We proceed as in Lemma 4.6 of Ref. 18. Indeed, given \( \zeta \in \hat{H}^{K}_h \) and \( \tau \in H^{K}_h \), we certainly have that \( \hat{\Pi}^K_h (\zeta) = \zeta \), and hence we deduce from (4.7) and (A.4) that
\[
a_h^K (\zeta, \tau) = \frac{1}{\mu} \int_K (\hat{\Pi}^K_h (\zeta))^d \cdot (\hat{\Pi}^K_h (\tau))^d + \frac{1}{\alpha} \int_K \text{div} (\zeta) \cdot \text{div} (\tau)
\]
\[
= \frac{1}{\mu} \int_K (\hat{\Pi}^K_h (\zeta))^d : \tau^d + \frac{1}{\alpha} \int_K \text{div} (\zeta) \cdot \text{div} (\tau)
\]
\[
= \frac{1}{\mu} \int_K \zeta^d : \tau^d + \frac{1}{\alpha} \int_K \text{div} (\zeta) \cdot \text{div} (\tau) = a^K (\zeta, \tau),
\]
which proves (4.8). Next, for the boundedness of \( a_h^K \) we apply the Cauchy–Schwarz inequality, the estimate (A.3), and the upper bound in (4.6), to obtain
\[
\| a_h^K (\zeta, \tau) \| \leq \frac{1}{\mu} \| (\hat{\Pi}^K_h (\zeta))^d \|_{0,K} \| (\hat{\Pi}^K_h (\tau))^d \|_{0,K} + \frac{1}{\alpha} \| \text{div} (\zeta) \|_{0,K} \| \text{div} (\tau) \|_{0,K}
\]
\[
+ \{ S^K (\zeta - \hat{\Pi}^K_h (\zeta), \zeta - \hat{\Pi}^K_h (\zeta)) \}^{1/2} \{ S^K (\tau - \hat{\Pi}^K_h (\tau), \tau - \hat{\Pi}^K_h (\tau)) \}^{1/2}
\]
\[
\leq \frac{1}{\mu} \hat{C}^2 \| \zeta \|_{\text{div},K} \| \tau \|_{\text{div},K} + \frac{1}{\alpha} \| \zeta \|_{\text{div},K} \| \tau \|_{\text{div},K}
\]
\[
+ c_1 \| \zeta - \hat{\Pi}^K_h (\zeta) \|_{0,K} \| \tau - \hat{\Pi}^K_h (\tau) \|_{0,K}
\]
\[
\leq \frac{1}{\mu} \hat{C}^2 \| \zeta \|_{\text{div},K} \| \tau \|_{\text{div},K} + \frac{1}{\alpha} \| \zeta \|_{\text{div},K} \| \tau \|_{\text{div},K}
\]
\[
+ c_1 (1 + \hat{C})^2 \| \zeta \|_{0,K} \| \tau \|_{0,K},
\]
for all \( \zeta, \tau \in \hat{H}^{K}_h \), which gives (4.9) with
\[
\alpha_1 := \max \left\{ \frac{1}{\mu} \hat{C}^2 + \frac{1}{\alpha}, c_1 (1 + \hat{C})^2 \right\}. \tag{4.11}
\]

Finally, concerning (4.10), it is clear that the corresponding upper bound follows from (4.9). In turn, applying the lower estimate in (4.6), we find that
\[
\| \zeta^d \|_{0,K}^2 + \| \text{div} (\zeta) \|_{0,K}^2
\]
\[
\leq 2 \| (\hat{\Pi}^K_h (\zeta))^d \|_{0,K}^2 + \| \text{div} (\zeta) \|_{0,K}^2 + 2 \| (\zeta - \hat{\Pi}^K_h (\zeta))^d \|_{0,K}^2
\]
\[
\leq 2 \mu \left( \frac{1}{\mu} \| (\hat{\Pi}^K_h (\zeta))^d \|_{0,K}^2 \right) + \alpha \left( \frac{1}{\alpha} \| \text{div} (\zeta) \|_{0,K}^2 \right) + \frac{2}{c_0} (c_0 \| \zeta - \hat{\Pi}^K_h (\zeta) \|_{0,K}^2)
\]
\[
\leq 2 \mu \left( \frac{1}{\mu} \| (\hat{\Pi}^K_h (\zeta))^d \|_{0,K}^2 \right) + \alpha \left( \frac{1}{\alpha} \| \text{div} (\zeta) \|_{0,K}^2 \right)
\]
\[
+ \frac{2}{c_0} S^K (\zeta - \hat{\Pi}^K_h (\zeta), \zeta - \hat{\Pi}^K_h (\zeta)),
\]
for all $\zeta \in \mathbb{H}_h^K$, which yields the lower bound in (4.10) with

$$\alpha_2 := \max \left\{ 2\mu, \alpha, \frac{2}{c_0} \right\}^{-1}. \quad (4.12)$$

4.4. Two particular choices for $\hat{\mathbb{H}}_h^K$ and $\hat{\Pi}_h^K$

We now proceed to define two possible choices for $\hat{\mathbb{H}}_h^K$ and the projection $\hat{\Pi}_h^K : \mathbb{H}(\text{div}; K) \rightarrow \hat{\mathbb{H}}_h^K$.

First choice

We first consider $\hat{\mathbb{H}}_h^K := \mathbb{P}_k(K)$, which clearly satisfies (A.1), and let $\hat{\Pi}_h^K := \mathcal{P}_k^K : L^2(K) \rightarrow \mathbb{P}_k(K)$ be the $L^2(K)$-orthogonal projection. In other words, $\mathcal{P}_k^K$ stands for the operator $\mathcal{P}_k^K$ (cf. 3.9) acting along each row of a tensor in $L^2(K)$. In turn, note that in the present 2D case there holds

$$\dim \hat{\mathbb{H}}_h^K = 4 \dim \mathbb{P}_k(K) = (k + 1)(2k + 4). \quad (4.13)$$

It follows straightforwardly from Sec. 3.3 that $\mathcal{P}_k^K$ satisfies the assumptions (A.2) and (A.3) with $\hat{C} = 1$, whereas (A.5) is simply the tensorial version of (3.11). For the remaining assumption we use the characterization of $\mathcal{P}_k^K$ and the fact that for each $\rho \in \mathbb{P}_k(K)$ there certainly holds $\rho^d \in \mathbb{P}_k(K)$. Thus, we find that

$$\int_K (\hat{\Pi}_h^K(\zeta))^d : (\hat{\Pi}_h^K(\tau))^d = \int_K \hat{\Pi}_h^K(\tau) : (\hat{\Pi}_h^K(\zeta))^d = \int_K \tau : (\hat{\Pi}_h^K(\zeta))^d$$

for all $\zeta, \tau \in \mathbb{H}_h^K$, which proves (A.4).

Second choice

On the other hand, because of the identity $\sigma^d = \mu \nabla u$, with $\text{div}(u) = 0$ in $\Omega$ (cf. (2.2)), we now suggest the same projection defined in Sec. 4 of Ref. 18 for the linear Stokes problem. More precisely, we consider the subspace of $\mathbb{P}_k(K)$ given by

$$\hat{\mathbb{H}}_h^K := \hat{\mathbb{H}}_h^K, \nabla \oplus \hat{\mathbb{H}}_h^K, \text{I}, \quad (4.14)$$

where

$$\hat{\mathbb{H}}_h^K, \nabla := \{ \nabla \text{curl}(q) : q \in \text{span}\{ x^\alpha : 2 \leq |\alpha| \leq k + 2 \} \subseteq \mathbb{P}_{k+2}(K) \}$$

and

$$\hat{\mathbb{H}}_h^K, \text{I} := \{ q^I : q \in \mathbb{P}_k(K) \}.$$ 

Then, we recall from Lemma 4.1 of Ref. 18 that there holds

$$\dim \hat{\mathbb{H}}_h^K = (k + 1)(k + 4). \quad (4.15)$$

In addition, it is easy to see from (4.14) that $\text{tr}(\tau) = 0$ for all $\tau \in \hat{\mathbb{H}}_h^K, \nabla$. 
In turn, as explained in Ref. 18, the corresponding projection operator \( \tilde{\Pi}_k^K : \mathbb{H}(\text{div}; K) \rightarrow \mathbb{H}_K^K \) is defined in terms of the decomposition:

\[
\tilde{\Pi}_k^K(\zeta) := \tilde{\zeta}_\nabla + q_\zeta \mathbb{I} + c_\zeta \mathbb{I} \in \mathbb{H}_K^K, 
\]

where the components \( \tilde{\zeta}_\nabla \in \mathbb{H}_K^K \), \( q_\zeta \in \tilde{P}_k(K) := \text{span}\{x^\alpha : 1 \leq |\alpha| \leq k\} \), and \( c_\zeta \in \mathbb{R} \) are computed according to the following sequentially connected problems:

- find \( \tilde{\zeta}_\nabla \in \mathbb{H}_K^K \) such that

\[
\int_K \tilde{\zeta}_\nabla : \tau = \int_K \zeta : \tau, \quad \forall \tau \in \mathbb{H}_K^K, 
\]

- find \( q_\zeta \in \tilde{P}_k(K) \) such that

\[
\int_K \text{div}(q_\zeta \mathbb{I}) \cdot \text{div}(q \mathbb{I}) = \int_K \text{div}(\zeta - \tilde{\zeta}_\nabla) \cdot \text{div}(q \mathbb{I}), \quad \forall q \in \tilde{P}_k(K), 
\]

and

- find \( c_\zeta \in \mathbb{R} \) such that

\[
\int_K \text{tr}(\tilde{\Pi}_k^K(\zeta)) = \int_K \text{tr}(\zeta). 
\]

Regarding the above definition of \( \tilde{\Pi}_k^K \), we first remark that the unique solvability of (4.17) was guaranteed in Sec. 4 of Ref. 18. In particular, note that having computed \( \tilde{\zeta}_\nabla \in \mathbb{H}_K^K \) and then \( q_\zeta \in \tilde{P}_k(K) \), the identity (4.17c) yields

\[
c_\zeta = \frac{1}{2|M_K|} \int_K \{\text{tr}(\zeta) - 2q_\zeta\}. 
\]

Next, we check that the right-hand sides of (4.17) are indeed calculable when \( \zeta \) belongs to our virtual space \( \mathbb{H}_K^K \) (cf. (3.5)). Firstly, for each \( \tau := \nabla \text{curl}(q) \in \mathbb{H}_K^K \) we have that

\[
\int_K \zeta : \tau = \int_K \text{curl}(q) = -\int_K \text{curl}(q) \cdot \text{div}(\zeta) + \int_{\partial K} \zeta n \cdot \text{curl}(q),
\]

which establishes that the right-hand side of (4.17a) can be explicitly computed for \( \zeta \in \mathbb{H}_K^K \). In turn, since \( \text{div}(\zeta) \in \tilde{P}_k(K) \) (cf. (3.5)) and \( \tilde{\zeta}_\nabla \in \mathbb{H}_K^K \), it is quite clear that the right-hand side of (4.17b) is also calculable for each \( q \in \tilde{P}_k(K) \). Finally, for the right-hand side of (4.17c) we simply observe that

\[
\int_K \text{tr}(\zeta) = \int_K \zeta : \mathbb{I} = \int_K \text{div}(\zeta) + \int_{\partial K} \zeta n \cdot x, 
\]

which, according to (3.5), is calculable as well. Note here that the above-described computation of \( \tilde{\Pi}_k^K(\zeta) \) does not make use of the degrees of freedom given by \( m_{r, \mathbb{K}} \) (cf. (3.6)).

Next, it is straightforward to check from (4.17) that \( \tilde{\Pi}_k^K(\zeta) = \zeta \) for all \( \zeta \in \mathbb{H}_K^K \), which confirms that \( \tilde{\Pi}_k^K \) is in fact a projector. Hence, from the above discussion we have that the second choice of \( \tilde{\Pi}_k^K \) satisfies the assumptions (A.1) and (A.2). For
and (A.5) we refer to Lemmas 4.2 and 4.4 of Ref. 18, respectively. Finally, employing (4.16) and (4.17a) we find that for all \( \zeta, \tau \in H(\text{div}; K) \) there holds
\[
\int_K (\hat{\Pi}_h^K(\zeta))^d : (\hat{\Pi}_h^K(\tau))^d = \int_K \hat{\zeta}_V : \hat{\tau}_V = \int_K \hat{\zeta}_V : \tau
\]
\[
= \int_K (\widehat{\Pi}_h^K(\zeta))^d : \tau = \int_K (\widehat{\Pi}_h^K(\zeta))^d : \tau^d,
\]
which constitutes (A.4).

5. The Mixed Virtual Element Scheme

According to the analysis from the foregoing sections, and given an integer \( k \geq 0 \), we now consider the virtual element subspace
\[
H_h := H_h^k \cap H_0(\text{div}; \Omega),
\]
where \( H_h^k \) is defined in (3.18) (or (3.19)). Next, as suggested by (4.7), we define the global discrete bilinear form \( a_h : H_h \times H_h \rightarrow \mathbb{R} \) as
\[
a_h(\zeta, \tau) := \sum_{K \in T_h} a_h^K(\zeta, \tau), \quad \forall \zeta, \tau \in H_h.
\]
Then, the Galerkin scheme associated with (2.5) reads: find \( \sigma_h \in H_h \) such that
\[
a_h(\sigma_h, \tau_h) = F(\tau_h) := -\frac{1}{\alpha} \int_{\Omega} f \cdot \text{div}(\tau_h) + \langle \tau_h n, g \rangle_{\Gamma}, \quad \forall \tau_h \in H_h.
\]

5.1. Solvability and a priori error analysis

The following result provides the discrete analogue of Lemma 2.2.

**Lemma 5.1.** There exists \( \eta > 0 \), independent of \( h \), such that
\[
a_h(\zeta_h, \zeta_h) \geq \eta \| \zeta_h \|^2_{\text{div}; \Omega}, \quad \forall \zeta_h \in H_h.
\]

**Proof.** We adapt the proofs of Lemma 5.2 of Ref. 18 and Lemma 2.2. Indeed, according to the definition of \( a_h \) (cf. (5.2)), we apply the lower bound in (4.10) and Lemma 2.1 to find that for each \( \zeta_h \in H_h \) there holds
\[
a_h(\zeta_h, \zeta_h) = \sum_{K \in T_h} a_h^K(\zeta_h, \tau_h)
\]
\[
\geq \alpha_2 \{ \| \zeta_h^d \|^2_{0, \Omega} + \| \text{div}(\zeta_h) \|^2_{0, \Omega} \}
\]
\[
\geq c_\Omega \frac{\alpha_2}{2} \| \zeta_h \|^2_{0, \Omega} + \frac{\alpha_2}{2} \| \text{div}(\zeta_h) \|^2_{0, \Omega},
\]
which yields (5.4) with
\[
\eta := \frac{\alpha_2}{2} \min\{1, c_\Omega\},
\]
thus finishing the proof. \( \square \)
The unique solvability and stability of the actual Galerkin scheme (5.3) is established now.

**Theorem 5.1.** There exists a unique $\sigma_h \in H_h$ solution of (5.3), and there exists a positive constant $C$, independent of $h$, such that

$$\|\sigma_h\|_{\text{div};\Omega} \leq C (\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}).$$

**Proof.** The boundedness of $a_h : H_h \times H_h \to \mathbb{R}$ with respect to the norm $\|\cdot\|_{\text{div};\Omega}$ of $H$ follows easily from (4.9). Hence Lemma 5.1 and a straightforward application of the Lax–Milgram lemma complete the proof.  

We now aim to derive the corresponding a priori error estimates for (5.3) and (2.5). For this purpose, we will make use of the global interpolation operator $\Pi^K : H^1(\Omega) \to H_h^K$ (cf. Sec. 3.4), whose local restriction is denoted $\Pi^K_K : H^1(\Omega) \to H_h^K$. In turn, given the local projector $\hat{\Pi}^K_K : H(\text{div};K) \to H_h^K$ defined by the assumptions (A.1)–(A.5), we denote by $\hat{\Pi}^K_K$ its global counterpart, that is we let

$$\hat{\Pi}^K_K(\zeta)|_K := \hat{\Pi}^K_K(\zeta)_K, \quad \forall K \in T_h, \quad \forall \zeta \in H.$$

Then, we have the following main result.

**Theorem 5.2.** Let $\sigma \in H$ and $\sigma_h \in H_h$ be the unique solutions of the continuous and discrete schemes (2.5) and (5.3), respectively, and assume that $\sigma \in H^1_0(\Omega)$. Then, there exists a positive constant $\bar{C}$, independent of $h$, such that

$$\|\sigma - \sigma_h\|_{\text{div};\Omega} \leq \bar{C} (\|\sigma - \Pi^K_0(\sigma)\|_{\text{div};\Omega} + \|\sigma - \hat{\Pi}^K_K(\sigma)\|_{0,\Omega}).$$

**Proof.** We adapt the proof of Theorem 5.2 of Ref. 18. Indeed, we first observe, thanks to the triangle inequality, that

$$\|\sigma - \sigma_h\|_{\text{div};\Omega} \leq \|\sigma - \Pi^K_0(\sigma)\|_{\text{div};\Omega} + \|\Pi^K_0(\sigma) - \sigma_h\|_{\text{div};\Omega},$$

whence it only remains to estimate $\delta_h := \Pi^K_0(\sigma) - \sigma_h \in H_h$. In fact, applying Lemma 5.1, adding and subtracting $\hat{\Pi}^K_K(\sigma)$, using the discrete and continuous formulations (5.3) and (2.5), respectively, employing (4.8), and finally utilizing the definitions of $a^K_h$ and $a^K$ (cf. (4.7), (4.1)), we find that

$$\eta^2 \|\delta_h\|^2_{\text{div};\Omega} \leq a_h(\delta_h, \delta_h) = a_h(\Pi^K_0(\sigma), \delta_h) - a_h(\sigma_h, \delta_h) \leq a_h(\Pi^K_0(\sigma) - \hat{\Pi}^K_K(\sigma), \delta_h) + a_h(\hat{\Pi}^K_K(\sigma), \delta_h) - a(\sigma, \delta_h) \leq \sum_{K \in T_h} \{a^K_h(\Pi^K_0(\sigma) - \hat{\Pi}^K_K(\sigma), \delta_h) + a^K(\hat{\Pi}^K_K(\sigma) - \sigma, \delta_h)\} = \sum_{K \in T_h} \left\{ \frac{1}{\mu} \int_K (\hat{\Pi}^K_K(\Pi^K_0(\sigma) - \sigma)) : (\hat{\Pi}^K_K(\delta_h)) \right\}. $$
there hold

\[ + \frac{1}{\mu} \int_K (\hat{\Pi}^K_\delta(\sigma) - \sigma)^d : \delta_h + \frac{1}{\alpha} \int_K \text{div}(\Pi^K_\delta(\sigma) - \sigma) \cdot \text{div}(\delta_h) \]

\[ + S^K(\Pi^K_\delta(\sigma) - \hat{\Pi}^K_\delta(\Pi^K_\delta(\sigma)), \delta_h - \hat{\Pi}^K_\delta(\delta_h)) \].

Next, using from (3.14) that \( \text{div}(\Pi^K_\delta(\sigma)) = P^K_\delta(\text{div}(\sigma)) \), where \( P^K_\delta : L^2(K) \to P_\delta(K) \) is the orthogonal projector, and recalling from the sequential definition of \( \mathbb{H}_h \) (cf. (3.1), (3.19), and (5.1)) that \( \text{div}(\delta_h) \in P_\delta(K) \), we realize that the first term in the last line of the foregoing equation vanishes for each \( K \in T_h \). Hence, applying now to the remaining terms the Cauchy–Schwarz inequality, the upper bound from (4.6), and the estimate provided by the assumption (A.3), we obtain, with the constant

\[ C_0 := \frac{1}{\eta} \max \left\{ \frac{1}{2 \mu}, \frac{1}{\mu}, c_1(1 + \hat{C}) \right\} \]

that

\[ \| \delta_h \|_{\text{div},\Omega}^2 \leq C_0 \left\{ \| \sigma - \Pi^h_\delta(\sigma) \|_{0,\Omega} + \| \sigma - \hat{\Pi}^h_\delta(\sigma) \|_{0,\Omega} \right. \]

\[ + \| \Pi^h_\delta(\sigma) - \hat{\Pi}^h_\delta(\Pi^h_\delta(\sigma)) \|_{0,\Omega} \| \delta_h \|_{\text{div},\Omega} , \]

which yields

\[ \| \delta_h \|_{\text{div},\Omega} \leq C_0 \left\{ \| \sigma - \Pi^h_\delta(\sigma) \|_{0,\Omega} + \| \sigma - \hat{\Pi}^h_\delta(\sigma) \|_{0,\Omega} + \| \Pi^h_\delta(\sigma) - \hat{\Pi}^h_\delta(\Pi^h_\delta(\sigma)) \|_{0,\Omega} \right\} . \]

(5.9)

Next, adding and subtracting \( \sigma - \hat{\Pi}^h_\delta(\sigma) \), and employing again the boundedness of \( \hat{\Pi}^h_\delta \) (cf. (A.3)), we deduce that

\[ \| \Pi^h_\delta(\sigma) - \hat{\Pi}^h_\delta(\Pi^h_\delta(\sigma)) \|_{0,\Omega} \]

\[ \leq \| \sigma - \Pi^h_\delta(\sigma) \|_{0,\Omega} + \| \sigma - \hat{\Pi}^h_\delta(\sigma) \|_{0,\Omega} + \| \hat{\Pi}^h_\delta(\sigma) - \Pi^h_\delta(\sigma) \|_{0,\Omega} \]

\[ \leq (1 + \hat{C}) \{ \| \sigma - \Pi^h_\delta(\sigma) \|_{\text{div},\Omega} + \| \sigma - \hat{\Pi}^h_\delta(\sigma) \|_{0,\Omega} \}, \]

which, together with (5.9) and (5.7), imply the estimate (5.6) and complete the proof.

Having established the \textit{a priori} error estimates for our unknown, we now provide the corresponding rate of convergence. Recall from (4.2) the definition of the space \( \mathbb{H}_h^{\text{curl}}(K) \).

**Theorem 5.3.** Let \( \sigma \in \mathbb{H} \) and \( \sigma_h \in \mathbb{H}_h \) be the unique solutions of the continuous and discrete schemes (2.5) and (5.3), respectively. Assume that for some \( r \in [1, k + 1] \) there hold \( \sigma|_K \in \mathbb{H}_h^{\text{curl}}(K) \subseteq \mathbb{H}^r(K) \) and \( \text{div}(\sigma)|_K \in H^r(K) \) for each \( K \in T_h \). Then, there exists a positive constant \( C \), independent of \( h \), such that

\[ \| \sigma - \sigma_h \|_{\text{div},\Omega} \leq Ch^r \sum_{K \in T_h} (|\sigma|_{r,K} + |\text{div}(\sigma)|_{r,K}) . \]

(5.10)
5.2. Computable approximations of $\sigma$, $p$, and $u$

We now introduce the fully computable approximation of $\sigma_h$ given by

$$\tilde{\sigma}_h := \tilde{\Pi}_h^k(\sigma_h),$$

(5.11)

and establishes next the corresponding a priori error estimate in the $L^2(\Omega)$-norm, which, as shown below in Theorem 5.4, yields exactly the same rate of convergence given by Theorem 5.3.

Lemma 5.2. There exists a positive constant $C$, independent of $h$, such that

$$\|\sigma - \tilde{\sigma}_h\|_{0,\Omega} \leq C \{ \|\sigma - \Pi_h^k(\sigma)\|_{\text{div},\Omega} + \|\sigma - \tilde{\Pi}_h^k(\sigma)\|_{0,\Omega} \}.\tag{5.12}$$

Proof. Similarly as in Theorem 5.4 of Ref. 18, we first write by triangle inequality that

$$\|\sigma - \tilde{\sigma}_h\|_{0,\Omega} \leq \|\sigma - \sigma_h\|_{0,\Omega} + \|\sigma_h - \tilde{\Pi}_h^k(\sigma_h)\|_{0,\Omega}.\tag{5.13}$$

Next, adding and subtracting $\sigma$ and $\tilde{\Pi}_h^k(\sigma)$, and utilizing the boundedness of $\tilde{\Pi}_h^K$ (cf. (A.3)), we find that

$$\|\sigma_h - \tilde{\Pi}_h^k(\sigma_h)\|_{0,\Omega} \leq \|\sigma - \sigma_h\|_{0,\Omega} + \|\sigma - \tilde{\Pi}_h^k(\sigma)\|_{0,\Omega} + \|\tilde{\Pi}_h^k(\sigma - \sigma_h)\|_{0,\Omega} \leq C \{ \|\sigma - \sigma_h\|_{\text{div},\Omega} + \|\sigma - \tilde{\Pi}_h^k(\sigma)\|_{0,\Omega} \},\tag{5.14}$$

which, replaced back into (5.13), and then combined with (5.6), gives (5.12) and completes the proof.

We remark here that the lack of the approximation property of $\tilde{\Pi}_h^K$ in the whole $H(\text{div};\Omega)$-norm and the fact that $\tilde{\sigma}_h$ does not necessarily belong to $H(\text{div};\Omega)$ do not allow us to establish an estimate for $\|\sigma - \tilde{\sigma}_h\|_{\text{div},\Omega}$. Nevertheless, thanks to a suitable postprocessing of $\tilde{\sigma}_h$, we are able to provide below (cf. Sec. 5.3) an explicitly
calculable second approximation of $\sigma$ yielding an optimal rate of convergence in the broken $H(div)$-norm.

On the other hand, since we know from (2.3) that $p = -\frac{1}{2} \text{tr}(\sigma)$, we now suggest to define the following approximation of the pressure:

$$p_h := -\frac{1}{2} \text{tr}(\sigma_h),$$  \hspace{1cm} (5.15)

so that there holds

$$\|p - p_h\|_\Omega \leq \frac{1}{\sqrt{2}} \|\sigma - \tilde{\sigma}\|_{0,\Omega},$$

which, together with (5.12), gives the $a$ priori error estimate for the pressure, that is

$$\|p - p_h\|_{0,\Omega} \leq C \{\|\sigma - \Pi_h^k(\sigma)\|_{\text{div},\Omega} + \|\sigma - \tilde{\Pi}_h^k(\sigma)\|_{0,\Omega}\}. \hspace{1cm} (5.16)$$

In turn, resembling (2.4), we now set the approximation of $u$ as

$$u_h := \frac{1}{\alpha} \{P_h^k(f) + \text{div}(\sigma_h)\}, \hspace{1cm} (5.17)$$

where $P_h^k$ is the $L^2(\Omega)$-orthogonal projector onto the space of piecewise polynomial vectors of degree $\leq k$. Equivalently, we can set $P_h^k(v)|_K = P_h^k(v)|_K$ for each $K \in T_h$, for all $v \in L^2(\Omega)$, where $P_h^k : L^2(K) \to P_h(K)$ is the local orthogonal projector defined in (3.9). Hence, it readily follows from (2.4) and (5.17) that

$$\|u - u_h\|_{0,\Omega} \leq \frac{1}{\alpha} \{\|f - P_h^k(f)\|_{0,\Omega} + \|\text{div}(\sigma - \sigma_h)\|_{0,\Omega}\},$$

from which, using that $f = au - \text{div}(\sigma)$ (cf. (2.1)), bounding $\|\text{div}(\sigma - \sigma_h)\|_{0,\Omega}$ by $\|\sigma - \sigma_h\|_{\text{div},\Omega}$, and employing the $a$ priori error estimate (5.6) (cf. Theorem 5.2), we deduce that

$$\|u - u_h\|_{0,\Omega} \leq C \{\|u - P_h^k(u)\|_{0,\Omega} + \|\text{div}(\sigma) - P_h^k(\text{div}(\sigma))\|_{0,\Omega} + \|\sigma - \Pi_h^k(\sigma)\|_{\text{div},\Omega} + \|\sigma - \tilde{\Pi}_h^k(\sigma)\|_{0,\Omega}\}. \hspace{1cm} (5.18)$$

In this way, we are now able to provide the theoretical rates of convergence for $\tilde{\sigma}_h, p_h$, and $u_h$.

**Theorem 5.4.** Let $\sigma \in H$ and $\sigma_h \in H_h$ be the unique solutions of the continuous and discrete schemes (2.5) and (5.3), respectively. In addition, let $\tilde{\sigma}_h, p_h$, and $u_h$ be the discrete approximations introduced in (5.11), (5.15), and (5.17), respectively. Assume that for some $r \in [1, k + 1]$ there hold $\sigma|_K \in H^r_c(K) \subseteq H^r(K), \text{div}(\sigma)|_K \in H^r(K)$, and $u|_K \in H^r(K)$ for each $K \in T_h$. Then, there exist positive constants $C_1$ and $C_2$, independent of $h$, such that

$$\|\sigma - \tilde{\sigma}_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq C_1 h^r \sum_{K \in T_h} \{|\sigma|_{r,K} + |\text{div}(\sigma)|_{r,K}\}. \hspace{1cm} (5.19)$$
and
\[ \|u - u_h\|_{0,\Omega} \leq C_2 h^r \sum_{K \in T_h} \{ |\sigma|_{r,K} + |\text{div}(\sigma)|_{r,K} + |u|_{r,K} \}. \] (5.20)

**Proof.** It follows from (5.12), (5.16), (5.18), and a straightforward application of the approximation properties provided by (3.11), (3.16), and (A.5). \( \square \)

### 5.3. A convergent approximation of \( \sigma \) in the broken \( \mathbb{H}(\text{div}) \)-norm

Finally, motivated by the approach developed in Refs. 23 and 24, we now construct a second approximation \( \sigma^*_h \) of the pseudostress \( \sigma \), which is shown below to yield an optimal rate of convergence in the broken \( \mathbb{H}(\text{div}) \)-norm. To this end, we first consider for each \( K \in T_h \) an arbitrary finite-dimensional subspace \( V(K) \) of \( \mathbb{H}(\text{div}; K) \), which is going to be suitably chosen later on. Then, we let \( (\cdot, \cdot)_{\text{div}; K} \) be the usual \( \mathbb{H}(\text{div}; K) \)-inner product with induced norm \( \| \cdot \|_{\text{div}; K} \), and set \( \sigma^*_h \mid K := \sigma^*_{h,K} \in V(K) \), where \( \sigma^*_{h,K} \) is the unique solution of the local problem
\[ (\sigma^*_{h,K}, \tau_h)_{\text{div}; K} = \int_K \hat{\sigma}_h : \tau_h + \int_K \text{div}(\sigma_h) \cdot \text{div}(\tau_h), \quad \forall \tau_h \in V(K). \] (5.21)

We emphasize that \( \sigma^*_{h,K} \) can be explicitly (and, if the definition of \( V(K) \) allows it, efficiently) calculated for each \( K \in T_h \), independently. Throughout the rest of the section we let \( \Pi^K_{\text{div}} : H(\text{div}; K) \to V(K) \) be the orthogonal projector with respect to \( (\cdot, \cdot)_{\text{div}; K} \). Then, the following result establishes the *a priori* estimate for the local error \( \| \sigma - \sigma^*_{h,K} \|_{\text{div}; K} \).

**Lemma 5.3.** For each \( K \in T_h \) there holds
\[ \| \sigma - \sigma^*_{h,K} \|_{\text{div}; K} \leq \| \sigma - \hat{\sigma}_h \|_{0,K} + \| \text{div}(\sigma - \sigma_h) \|_{0,K} + \| \sigma - \Pi^K_{\text{div}}(\sigma) \|_{\text{div}; K}. \] (5.22)

**Proof.** We proceed as in the proof of Lemma 3.1 of Ref. 23. In fact, using the orthogonality condition \( (\sigma - \Pi^K_{\text{div}}(\sigma), \tau_h)_{\text{div}; K} = 0 \forall \tau_h \in V(K) \), together with (5.21), we first obtain the error equation
\[ (\Pi^K_{\text{div}}(\sigma) - \sigma^*_{h,K}, \tau_h)_{\text{div}; K} = \int_K (\sigma - \hat{\sigma}_h) : \tau_h \]
\[ + \int_K \text{div}(\sigma - \sigma_h) \cdot \text{div}(\tau_h), \quad \forall \tau_h \in V(K). \]

Then, taking in particular \( \tau_h := \Pi^K_{\text{div}}(\sigma) - \sigma^*_{h,K} \in V(K) \) in the foregoing identity, and then using the Cauchy–Schwarz and triangle inequalities, we arrive at (5.22), thus finishing the proof. \( \square \)

At this point we notice from our previous analysis that the first two terms on the right-hand side of (5.22) converge at most with \( O(h^{k+1}) \), and hence we must choose \( V(K) \) so that at least this rate of convergence is guaranteed by the projection...
error $\|\sigma - \Pi_h^K(\sigma)\|_{\text{div},K}$ as well. According to it, and for simplicity, we now pick $V(K) := P_{k+1}(K)$, whence the resulting rate of convergence is given as follows.

**Theorem 5.5.** Let $\sigma \in H$ and $\sigma_h \in H_h$ be the unique solutions of the continuous and discrete schemes (2.5) and (5.3), respectively. In addition, let $\tilde{\sigma}_h$ and $\sigma_h^*$ be the discrete approximations introduced in (5.11) and (5.21) with $V(K) := P_{k+1}(K)$, respectively. Assume that for some $r \in [1, k + 1]$ there hold $\sigma|_K \in H^r_\text{curl}(K) \subseteq H^r(K)$ and $\text{div}(\sigma)|_K \in H^r(K)$ for each $K \in T_h$. Then, there exists a positive constant $C$, independent of $h$, such that

$$\left\{ \sum_{K \in T_h} \|\sigma - \sigma_h^*\|_{\text{div},K}^2 \right\}^{1/2} \leq Ch^{r} \sum_{K \in T_h} \{|\sigma|_{r,K} + |\text{div}(\sigma)|_{r,K}\}. \quad (5.23)$$

**Proof.** Given $K \in T_h$, we first let $P_{k+1}^K : \mathbb{L}^2(K) \to P_{k+1}(K)$ be the $L^2(K)$-orthogonal projector. Then, it readily follows that

$$\|\sigma - \Pi_h^K(\sigma)\|_{\text{div},K} \leq \|\sigma - P_{k+1}^K(\sigma)\|_{\text{div},K} \leq C\|\sigma - \tilde{\sigma}_h\|_{0,K} + \|\sigma - P_{k+1}^K(\sigma)\|_{1,K},$$

which, applying Lemma 3.4 of Ref. 18, leads to $\|\sigma - \Pi_h^K(\sigma)\|_{\text{div},K} \leq C h^r |\sigma|_{r,K}$. Next, replacing this estimate back into (5.22), summing the squares of the resulting (5.22) overall $K \in T_h$, and employing the upper bounds for $\|\sigma - \tilde{\sigma}_h\|_{0,K}$ and $\|\sigma - \sigma_h^*\|_{\text{div},\Omega}$ provided by (5.19) (cf. Theorem 5.4) and (5.10) (cf. Theorem 5.3), respectively, we conclude (5.23) and end the proof.

6. Numerical Results

In this section we present two numerical examples illustrating the good performance of the mixed virtual finite element scheme (5.3) introduced and analyzed in Sec. 5. Here, we use both choices of $\tilde{H}_h^K$ and $\tilde{H}_h^K$ described in Sec. 4.4. More precisely, we utilize the $L^2$-orthogonal projection and the projection defined by (4.17), which, according to the last name initials of the authors of Ref. 18, is called from now on the CG-projection. In turn, for all the computations we consider the virtual element subspace $H_h$ given by (5.1) with $k \in \{0, 1, 2\}$. However, similarly as in Ref. 19, the zero integral mean condition for tensors in the space $H_h$ is imposed via a real Lagrange multiplier. This means that (5.3) is reformulated, equivalently, as:

$$a_h(\sigma_h, \tau) + \lambda \int_{\Omega} \text{tr}(\tau) = F(\tau), \quad \forall \tau \in H_h^0,$$

$$\xi \int_{\Omega} \text{tr}(\sigma) = 0, \quad \forall \xi \in R. \quad (6.1)$$

Note here that the constraint $\int_{\Omega} \text{tr}(\sigma) = 0$, which is automatically satisfied by the members of the subspace $H_h$ (cf. (5.1)), is not incorporated in the definition.
of the space $H_h^k$ where $\sigma_h$ is sought now, but it is imposed weakly through the second equation of (6.1). In other words, $\lambda$ is an artificial unknown acting as the Lagrange multiplier taking care of that condition, which, thanks to the compatibility assumption satisfied by $g$, is known in advance to be 0. Nevertheless, $\lambda$ is kept in (6.1) to guarantee the symmetry of this equivalent system. On the other hand, concerning the polygonal decompositions of $\Omega$ employed in our computations, we consider uniform triangles as well as distorted squares and hexagons (the latter being generated by PolyMesher$^{31}$ in Example 1).

Furthermore, in what follows, $N$ stands for the total number of unknowns (d.o.f.) of (5.3), that is,

$$N := 2(k+1) \times \{\text{number of edges } e \in T_h\} + 2k(k+2)$$

$$\times \{\text{number of elements } K \in T_h\} + 1,$$

whereas the individual errors are defined by

$$e(\sigma) := \|\sigma - \tilde{\sigma}_h\|_{0, \Omega}, \quad e(p) := \|p - p_h\|_{0, \Omega}, \quad e(u) := \|u - u_h\|_{0, \Omega},$$

and

$$e(\sigma^*) := \left\{ \sum_{K \in T_h} \|\sigma - \sigma_{h,K}^*\|_{H^1(K)}^2 \right\}^{1/2},$$

where $\tilde{\sigma}_h$, $p_h$, $u_h$, and $\sigma_{h,K}^*$ are computed according to (5.11), (5.15), (5.17), and (5.21), respectively. In turn, the associated experimental rates of convergence are given by

$$r(\cdot) := \frac{\log(e(\cdot)/e'(\cdot))}{\log(h/h')},$$

where $e$ and $e'$ denote the corresponding errors for two consecutive meshes with sizes $h$ and $h'$, respectively. The numerical results presented below were obtained using a MATLAB code, where the corresponding linear systems were solved using its instruction "$\backslash$" as main solver.

In Example 1 we consider $\Omega := (-0.5, 1.5) \times (0, 2)$, $\mu = \alpha = 0.1$, and choose the data $f$ and $g$ so that the exact solution is given by the flow from Ref. 29, that is,

$$u(x) = \begin{pmatrix} 1 - \exp(\lambda x_1) \cos(2\pi x_2) \\ \frac{\lambda}{2\pi} \exp(\lambda x_1) \sin(2\pi x_2) \end{pmatrix}$$

and

$$p(x) = \frac{1}{2} \exp(2\lambda x_1) - \frac{1}{8\lambda} \{\exp(3\lambda) - \exp(-\lambda)\},$$

for all $x := (x_1, x_2) \in \Omega$, where $\lambda := \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}$ and $Re := \mu^{-1} = 10$ is the Reynolds number. Then, in Tables 1–3 we summarize the convergence history of the mixed virtual element scheme (5.3) as applied to Example 1, using
Table 1. Example 1, refinement with triangles and using the $L^2$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$r(\sigma)$</th>
<th>$e(u)$</th>
<th>$r(u)$</th>
<th>$e(p)$</th>
<th>$r(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$r(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2000</td>
<td>1241</td>
<td>1.53e-0</td>
<td>—</td>
<td>6.24e-1</td>
<td>8.51e-1</td>
<td>—</td>
<td>5.28e-0</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1000</td>
<td>4881</td>
<td>7.95e-1</td>
<td>0.94</td>
<td>2.61e-1</td>
<td>1.26</td>
<td>4.43e-1</td>
<td>0.94</td>
<td>2.74e-0</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>0.0500</td>
<td>19361</td>
<td>4.01e-1</td>
<td>0.99</td>
<td>1.22e-1</td>
<td>1.09</td>
<td>2.23e-1</td>
<td>0.99</td>
<td>1.38e-0</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>0.0333</td>
<td>43441</td>
<td>2.68e-1</td>
<td>1.00</td>
<td>8.04e-2</td>
<td>1.03</td>
<td>1.49e-1</td>
<td>1.00</td>
<td>9.25e-1</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>0.0250</td>
<td>77121</td>
<td>2.01e-1</td>
<td>1.00</td>
<td>6.00e-2</td>
<td>1.02</td>
<td>1.12e-1</td>
<td>1.00</td>
<td>6.94e-1</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>0.2000</td>
<td>4881</td>
<td>1.54e-1</td>
<td>—</td>
<td>6.03e-2</td>
<td>9.93e-2</td>
<td>—</td>
<td>6.02e-1</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1000</td>
<td>19361</td>
<td>4.13e-2</td>
<td>1.90</td>
<td>1.49e-2</td>
<td>2.02</td>
<td>2.64e-2</td>
<td>1.91</td>
<td>1.59e-1</td>
<td>1.92</td>
<td></td>
</tr>
<tr>
<td>0.0500</td>
<td>77121</td>
<td>1.07e-2</td>
<td>1.95</td>
<td>3.69e-3</td>
<td>2.01</td>
<td>6.71e-3</td>
<td>1.98</td>
<td>4.04e-2</td>
<td>1.98</td>
<td></td>
</tr>
<tr>
<td>0.0333</td>
<td>173281</td>
<td>2.68e-1</td>
<td>1.96</td>
<td>1.64e-3</td>
<td>2.00</td>
<td>2.99e-3</td>
<td>1.99</td>
<td>1.80e-1</td>
<td>1.99</td>
<td></td>
</tr>
<tr>
<td>0.0250</td>
<td>307841</td>
<td>2.01e-1</td>
<td>1.97</td>
<td>9.22e-4</td>
<td>2.00</td>
<td>1.69e-3</td>
<td>1.99</td>
<td>1.01e-2</td>
<td>1.99</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Example 1, refinement with quadrilaterals and using the $L^2$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$r(\sigma)$</th>
<th>$e(u)$</th>
<th>$r(u)$</th>
<th>$e(p)$</th>
<th>$r(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$r(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2336</td>
<td>1681</td>
<td>1.38e-0</td>
<td>—</td>
<td>5.27e-1</td>
<td>9.33e-1</td>
<td>—</td>
<td>5.83e-0</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1161</td>
<td>5625</td>
<td>7.18e-1</td>
<td>0.93</td>
<td>2.48e-1</td>
<td>4.91e-1</td>
<td>0.92</td>
<td>3.11e-0</td>
<td>0.90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0580</td>
<td>22201</td>
<td>3.54e-1</td>
<td>1.10</td>
<td>2.29e-1</td>
<td>1.10</td>
<td>1.46e-0</td>
<td>1.09</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0387</td>
<td>49729</td>
<td>2.15e-1</td>
<td>1.09</td>
<td>7.59e-2</td>
<td>1.04</td>
<td>1.47e-1</td>
<td>1.09</td>
<td>9.39e-1</td>
<td>1.09</td>
<td></td>
</tr>
<tr>
<td>0.0290</td>
<td>88209</td>
<td>1.58e-1</td>
<td>1.07</td>
<td>5.66e-2</td>
<td>1.02</td>
<td>1.08e-1</td>
<td>1.07</td>
<td>6.90e-1</td>
<td>1.07</td>
<td></td>
</tr>
<tr>
<td>0.2336</td>
<td>5761</td>
<td>1.58e-1</td>
<td>—</td>
<td>6.80e-2</td>
<td>1.06e-1</td>
<td>—</td>
<td>6.75e-1</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1161</td>
<td>19463</td>
<td>4.35e-2</td>
<td>1.85</td>
<td>1.69e-2</td>
<td>1.99</td>
<td>2.96e-2</td>
<td>1.83</td>
<td>1.89e-1</td>
<td>1.82</td>
<td></td>
</tr>
<tr>
<td>0.0580</td>
<td>77257</td>
<td>9.49e-3</td>
<td>2.20</td>
<td>3.74e-3</td>
<td>2.18</td>
<td>6.46e-3</td>
<td>2.19</td>
<td>4.14e-2</td>
<td>2.19</td>
<td></td>
</tr>
<tr>
<td>0.0387</td>
<td>173383</td>
<td>3.88e-3</td>
<td>2.21</td>
<td>1.59e-2</td>
<td>2.11</td>
<td>2.63e-3</td>
<td>2.21</td>
<td>1.69e-2</td>
<td>2.21</td>
<td></td>
</tr>
<tr>
<td>0.0290</td>
<td>307841</td>
<td>2.07e-3</td>
<td>2.19</td>
<td>8.75e-4</td>
<td>2.08</td>
<td>1.40e-3</td>
<td>2.20</td>
<td>9.03e-3</td>
<td>2.19</td>
<td></td>
</tr>
<tr>
<td>0.2336</td>
<td>11441</td>
<td>1.79e-2</td>
<td>—</td>
<td>6.39e-3</td>
<td>1.20e-2</td>
<td>—</td>
<td>5.53e-2</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1161</td>
<td>38777</td>
<td>2.28e-3</td>
<td>2.95</td>
<td>7.82e-4</td>
<td>3.00</td>
<td>1.55e-3</td>
<td>2.93</td>
<td>8.32e-3</td>
<td>2.71</td>
<td></td>
</tr>
<tr>
<td>0.0580</td>
<td>154217</td>
<td>2.21e-4</td>
<td>3.36</td>
<td>8.13e-5</td>
<td>3.26</td>
<td>1.50e-4</td>
<td>3.36</td>
<td>8.81e-4</td>
<td>3.24</td>
<td></td>
</tr>
<tr>
<td>0.0387</td>
<td>346321</td>
<td>3.70e-5</td>
<td>3.35</td>
<td>2.25e-5</td>
<td>3.17</td>
<td>3.85e-5</td>
<td>3.36</td>
<td>2.32e-4</td>
<td>3.29</td>
<td></td>
</tr>
<tr>
<td>0.0290</td>
<td>615089</td>
<td>2.18e-5</td>
<td>3.33</td>
<td>9.17e-6</td>
<td>3.12</td>
<td>1.47e-5</td>
<td>3.35</td>
<td>8.99e-5</td>
<td>3.29</td>
<td></td>
</tr>
</tbody>
</table>

The L$^2$-projection. We notice that the rate of convergence of $O(h^{k+1})$ predicted by Theorems 5.4 and 5.5 (when $r = k + 1$) is attained for all the unknowns of this smooth example, for triangular as well as for quadrilateral and hexagonal meshes. In particular, these results confirm that our postprocessed pseudostress $\sigma^h$ improves in one power the unsatisfactory order provided by the first approximation $\tilde{\sigma}_h$ with respect to the broken $\mathbb{H}(\text{div})$-norm. In turn, in Tables 4–6 we present the convergence history of this example when using the CG-projection instead. Note that basically the same results were obtained with this projection. On the other hand, in order to illustrate the accurateness of the discrete scheme, in Figs. 1
and 2 we display some components of the approximate solutions for $k=2$ and the second meshes of each decomposition, using only the $L^2$-projection (no difference are observed with respect to the $CG$-projection).

In Example 2 we follow Ref. 18, and consider the L-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]^2$, $\mu = 1$, $\alpha = 0.5$, and choose the data $f$ and $g$ so that the exact solution is given by

$$u(x) = \begin{pmatrix} x_2^2 \\ -x_1^2 \end{pmatrix} \quad \text{and} \quad p(x) = (x_1^2 + x_2^2)^{1/3} - p_0,$$

### Table 3. Example 1, refinement with hexagons and using the $L^2$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$x(\sigma)$</th>
<th>$e(u)$</th>
<th>$x(u)$</th>
<th>$e(p)$</th>
<th>$x(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$x(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0930</td>
<td>6003</td>
<td>6.72e-1</td>
<td>2.66e-1</td>
<td>4.56e-1</td>
<td>2.92e-0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0738</td>
<td>10203</td>
<td>5.17e-1</td>
<td>1.13</td>
<td>2.00e-1</td>
<td>1.24</td>
<td>3.52e-1</td>
<td>1.12</td>
<td>2.25e-0</td>
<td>1.13</td>
<td></td>
</tr>
<tr>
<td>0.0556</td>
<td>18003</td>
<td>3.90e-1</td>
<td>1.00</td>
<td>1.53e-1</td>
<td>0.94</td>
<td>2.65e-1</td>
<td>1.00</td>
<td>1.70e-0</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>0.0400</td>
<td>36003</td>
<td>2.75e-1</td>
<td>1.05</td>
<td>1.06e-1</td>
<td>1.11</td>
<td>1.87e-1</td>
<td>1.05</td>
<td>1.20e-0</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>0.0294</td>
<td>66003</td>
<td>2.02e-1</td>
<td>1.01</td>
<td>7.78e-2</td>
<td>1.01</td>
<td>1.37e-1</td>
<td>1.01</td>
<td>8.82e-1</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>0.0930</td>
<td>18005</td>
<td>3.71e-2</td>
<td>1.85e-2</td>
<td>2.45e-2</td>
<td>1.60e-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0738</td>
<td>30605</td>
<td>2.20e-2</td>
<td>2.28</td>
<td>1.06e-2</td>
<td>2.43</td>
<td>1.46e-2</td>
<td>2.25</td>
<td>9.44e-2</td>
<td>2.27</td>
<td></td>
</tr>
<tr>
<td>0.0556</td>
<td>54005</td>
<td>1.26e-2</td>
<td>1.95</td>
<td>6.18e-3</td>
<td>1.96</td>
<td>8.38e-3</td>
<td>1.96</td>
<td>5.46e-2</td>
<td>1.93</td>
<td></td>
</tr>
<tr>
<td>0.0400</td>
<td>108005</td>
<td>6.27e-3</td>
<td>2.12</td>
<td>2.99e-3</td>
<td>2.13</td>
<td>4.17e-3</td>
<td>2.11</td>
<td>2.71e-2</td>
<td>2.11</td>
<td></td>
</tr>
<tr>
<td>0.0294</td>
<td>198005</td>
<td>3.38e-3</td>
<td>2.01</td>
<td>1.62e-3</td>
<td>2.01</td>
<td>2.25e-3</td>
<td>2.01</td>
<td>1.47e-2</td>
<td>2.00</td>
<td></td>
</tr>
</tbody>
</table>

### Table 4. Example 1, refinement with triangles and using the $CG$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$x(\sigma)$</th>
<th>$e(u)$</th>
<th>$x(u)$</th>
<th>$e(p)$</th>
<th>$x(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$x(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2000</td>
<td>1241</td>
<td>1.53e-0</td>
<td>6.24e-1</td>
<td>8.51e-1</td>
<td>5.28e-0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1000</td>
<td>4881</td>
<td>7.95e-1</td>
<td>0.94</td>
<td>2.61e-1</td>
<td>1.26</td>
<td>4.43e-1</td>
<td>0.94</td>
<td>2.74e-0</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>0.0500</td>
<td>19361</td>
<td>4.01e-1</td>
<td>0.99</td>
<td>1.22e-1</td>
<td>1.09</td>
<td>2.23e-1</td>
<td>0.99</td>
<td>1.38e-0</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>0.0333</td>
<td>43441</td>
<td>2.68e-1</td>
<td>1.00</td>
<td>8.04e-2</td>
<td>1.03</td>
<td>1.49e-1</td>
<td>1.00</td>
<td>9.25e-1</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>0.0250</td>
<td>77121</td>
<td>2.01e-1</td>
<td>1.00</td>
<td>6.00e-2</td>
<td>1.02</td>
<td>1.12e-1</td>
<td>1.00</td>
<td>8.94e-1</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>0.2000</td>
<td>4881</td>
<td>1.56e-1</td>
<td>6.03e-2</td>
<td>1.00e-1</td>
<td>6.02e-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1000</td>
<td>19361</td>
<td>4.17e-2</td>
<td>1.90</td>
<td>1.49e-2</td>
<td>2.02</td>
<td>2.66e-2</td>
<td>1.91</td>
<td>1.59e-1</td>
<td>1.92</td>
<td></td>
</tr>
<tr>
<td>0.0500</td>
<td>19361</td>
<td>4.17e-2</td>
<td>1.90</td>
<td>1.49e-2</td>
<td>2.02</td>
<td>2.66e-2</td>
<td>1.91</td>
<td>1.59e-1</td>
<td>1.92</td>
<td></td>
</tr>
<tr>
<td>0.0333</td>
<td>173281</td>
<td>4.86e-3</td>
<td>1.64e-3</td>
<td>2.00</td>
<td>3.01e-3</td>
<td>1.99</td>
<td>1.80e-2</td>
<td>1.99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0250</td>
<td>360721</td>
<td>2.76e-3</td>
<td>1.97</td>
<td>9.22e-4</td>
<td>2.00</td>
<td>1.69e-3</td>
<td>2.00</td>
<td>1.02e-2</td>
<td>1.99</td>
<td></td>
</tr>
<tr>
<td>0.0200</td>
<td>10121</td>
<td>2.10e-2</td>
<td>5.39e-3</td>
<td>2.01</td>
<td>1.40e-2</td>
<td>5.35e-2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0100</td>
<td>40241</td>
<td>2.76e-3</td>
<td>2.93</td>
<td>6.55e-4</td>
<td>3.04</td>
<td>1.85e-3</td>
<td>2.92</td>
<td>7.10e-3</td>
<td>2.91</td>
<td></td>
</tr>
<tr>
<td>0.0500</td>
<td>160481</td>
<td>3.51e-4</td>
<td>2.98</td>
<td>8.12e-5</td>
<td>3.01</td>
<td>2.35e-4</td>
<td>2.97</td>
<td>9.02e-4</td>
<td>2.98</td>
<td></td>
</tr>
<tr>
<td>0.0333</td>
<td>360721</td>
<td>1.04e-4</td>
<td>2.99</td>
<td>2.40e-5</td>
<td>3.00</td>
<td>7.00e-5</td>
<td>2.99</td>
<td>4.36e-4</td>
<td>3.19</td>
<td></td>
</tr>
<tr>
<td>0.0250</td>
<td>640961</td>
<td>4.42e-5</td>
<td>3.03</td>
<td>3.18e-5</td>
<td>3.02</td>
<td>3.18e-5</td>
<td>3.03</td>
<td>1.74e-4</td>
<td>2.99</td>
<td></td>
</tr>
</tbody>
</table>
Table 5. Example 1, refinement with quadrilaterals and using the CG-projection.

<table>
<thead>
<tr>
<th>k</th>
<th>h</th>
<th>N</th>
<th>e(σ)</th>
<th>r(σ)</th>
<th>e(u)</th>
<th>r(u)</th>
<th>e(p)</th>
<th>r(p)</th>
<th>e(σ*)</th>
<th>r(σ*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2336</td>
<td>1681</td>
<td>1.38e-0</td>
<td>—</td>
<td>5.27e-1</td>
<td>—</td>
<td>9.33e-1</td>
<td>—</td>
<td>5.83e-0</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.1161</td>
<td>5625</td>
<td>7.16e-1</td>
<td>0.93</td>
<td>2.48e-1</td>
<td>1.08</td>
<td>4.91e-1</td>
<td>0.92</td>
<td>3.11e-0</td>
<td>0.90</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0580</td>
<td>22201</td>
<td>3.34e-1</td>
<td>1.10</td>
<td>1.16e-1</td>
<td>1.10</td>
<td>2.29e-1</td>
<td>1.10</td>
<td>1.46e-0</td>
<td>1.09</td>
</tr>
<tr>
<td>0.0387</td>
<td>69729</td>
<td>2.15e-1</td>
<td>1.09</td>
<td>7.59e-2</td>
<td>1.04</td>
<td>1.47e-1</td>
<td>1.09</td>
<td>9.39e-1</td>
<td>1.09</td>
<td></td>
</tr>
<tr>
<td>0.0290</td>
<td>88209</td>
<td>1.58e-1</td>
<td>1.07</td>
<td>5.66e-2</td>
<td>1.02</td>
<td>1.08e-1</td>
<td>1.07</td>
<td>6.90e-1</td>
<td>1.07</td>
<td></td>
</tr>
<tr>
<td>0.2336</td>
<td>5761</td>
<td>1.59e-1</td>
<td>—</td>
<td>6.80e-2</td>
<td>—</td>
<td>1.07e-1</td>
<td>—</td>
<td>6.75e-1</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.1161</td>
<td>19463</td>
<td>4.36e-2</td>
<td>1.85</td>
<td>1.69e-2</td>
<td>1.99</td>
<td>2.96e-2</td>
<td>1.83</td>
<td>1.89e-1</td>
<td>1.82</td>
<td></td>
</tr>
</tbody>
</table>

Table 6. Example 1, refinement with hexagons and using the CG-projection.

<table>
<thead>
<tr>
<th>k</th>
<th>h</th>
<th>N</th>
<th>e(σ)</th>
<th>r(σ)</th>
<th>e(u)</th>
<th>r(u)</th>
<th>e(p)</th>
<th>r(p)</th>
<th>e(σ*)</th>
<th>r(σ*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0930</td>
<td>6003</td>
<td>6.72e-1</td>
<td>—</td>
<td>2.66e-1</td>
<td>—</td>
<td>4.56e-1</td>
<td>—</td>
<td>2.92e-0</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0738</td>
<td>10203</td>
<td>5.17e-1</td>
<td>1.13</td>
<td>2.00e-1</td>
<td>1.24</td>
<td>3.52e-1</td>
<td>1.12</td>
<td>2.25e-0</td>
<td>1.13</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0556</td>
<td>18003</td>
<td>3.90e-1</td>
<td>1.00</td>
<td>1.53e-1</td>
<td>0.94</td>
<td>2.65e-1</td>
<td>1.00</td>
<td>1.70e-0</td>
<td>0.99</td>
</tr>
<tr>
<td>0.0400</td>
<td>36003</td>
<td>2.75e-1</td>
<td>1.05</td>
<td>1.06e-1</td>
<td>1.11</td>
<td>1.87e-1</td>
<td>1.05</td>
<td>1.25e-0</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>0.0294</td>
<td>66003</td>
<td>2.02e-1</td>
<td>1.01</td>
<td>7.78e-2</td>
<td>1.01</td>
<td>1.37e-1</td>
<td>1.01</td>
<td>8.82e-1</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>0.0930</td>
<td>18005</td>
<td>3.72e-2</td>
<td>—</td>
<td>1.85e-2</td>
<td>—</td>
<td>2.46e-2</td>
<td>—</td>
<td>1.60e-1</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0738</td>
<td>30605</td>
<td>2.20e-2</td>
<td>2.28</td>
<td>1.06e-2</td>
<td>2.43</td>
<td>1.46e-2</td>
<td>2.25</td>
<td>9.44e-2</td>
<td>2.27</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0556</td>
<td>54005</td>
<td>1.27e-2</td>
<td>1.95</td>
<td>6.18e-3</td>
<td>1.90</td>
<td>8.39e-3</td>
<td>1.96</td>
<td>5.46e-2</td>
<td>1.93</td>
</tr>
<tr>
<td>0.0400</td>
<td>108005</td>
<td>6.28e-3</td>
<td>2.12</td>
<td>2.99e-3</td>
<td>2.19</td>
<td>4.18e-3</td>
<td>2.11</td>
<td>2.72e-2</td>
<td>2.11</td>
<td></td>
</tr>
<tr>
<td>0.0294</td>
<td>198005</td>
<td>3.39e-3</td>
<td>2.01</td>
<td>1.62e-3</td>
<td>2.01</td>
<td>2.25e-3</td>
<td>2.01</td>
<td>1.47e-2</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>0.0930</td>
<td>34007</td>
<td>2.26e-3</td>
<td>—</td>
<td>9.04e-4</td>
<td>—</td>
<td>1.49e-3</td>
<td>—</td>
<td>6.34e-3</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0738</td>
<td>57807</td>
<td>1.02e-3</td>
<td>3.46</td>
<td>3.88e-4</td>
<td>3.67</td>
<td>6.75e-4</td>
<td>3.44</td>
<td>2.86e-3</td>
<td>3.45</td>
<td></td>
</tr>
<tr>
<td>0.0556</td>
<td>102007</td>
<td>4.27e-4</td>
<td>3.07</td>
<td>1.78e-4</td>
<td>2.76</td>
<td>2.81e-4</td>
<td>3.10</td>
<td>1.27e-3</td>
<td>2.87</td>
<td></td>
</tr>
<tr>
<td>0.0400</td>
<td>204007</td>
<td>1.47e-4</td>
<td>3.22</td>
<td>5.93e-5</td>
<td>3.32</td>
<td>9.69e-5</td>
<td>3.22</td>
<td>4.42e-4</td>
<td>3.19</td>
<td></td>
</tr>
<tr>
<td>0.0294</td>
<td>374007</td>
<td>5.71e-5</td>
<td>3.08</td>
<td>2.35e-5</td>
<td>3.02</td>
<td>3.75e-5</td>
<td>3.10</td>
<td>1.76e-4</td>
<td>3.00</td>
<td></td>
</tr>
</tbody>
</table>

for all \( x := (x_1, x_2)^T \in \Omega \), where \( p_0 \in \mathbb{R} \) is such that \( \int_\Omega p = 0 \) holds. Note in this example that the partial derivatives of \( p \), and hence, in particular \( \text{div}(\sigma) \), are singular at the origin. More precisely, because of the power \( 1/3 \), there holds \( \sigma \in H^{5/3-\epsilon}(\Omega) \) and \( \text{div}(\sigma) \in H^{2/3-\epsilon}(\Omega) \) for each \( \epsilon > 0 \). Then, in Tables 7–12 we display the corresponding convergence history of Example 2, using again the two projections introduced in Sec. 4.4. As predicted by the theory, and due to the limited regularity of \( p \) and \( \sigma \) in this case, we observe that the orders \( O(h^{\min(k+1,5/3)}) \) and \( O(h^{2/3}) \) are attained by \( (\hat{\sigma}_h, \hat{p}_h) \) and \( \sigma^*_h \), respectively. In addition, we notice that \( u_h \) shows a convergence rate of \( O(h^{\min(k,5/3)+1}) \). This behavior of the error \( \|u - u_h\|_{0,\Omega} \) is explained by the fact that, as shown by (5.18), it depends on the regularity of \( u \),
Fig. 1. Example 1, $\sigma_{h,11}$ (top), $\sigma_{h,12}$ (center) and $u_{h,1}$ (bottom), using $k = 2$ and the second mesh of each kind (columns).
Fig. 2. Example 1, $\sigma_{h,11}^\star$ (top), $\sigma_{h,12}^\star$ (center) and $p_h$ (bottom), using $k = 2$ and the second mesh of each kind (columns).
Mixed virtual element method for Brinkman

Table 7. Example 2, refinement with triangles and using the $L^2$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$\gamma(\sigma)$</th>
<th>$e(u)$</th>
<th>$\gamma(u)$</th>
<th>$e(p)$</th>
<th>$\gamma(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$\gamma(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1667</td>
<td>1345</td>
<td>1.70e-1</td>
<td>—</td>
<td>7.89e-2</td>
<td>—</td>
<td>5.47e-2</td>
<td>—</td>
<td>1.95e-1</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0833</td>
<td>5281</td>
<td>8.45e-2</td>
<td>1.01</td>
<td>3.93e-2</td>
<td>1.01</td>
<td>2.59e-2</td>
<td>1.08</td>
<td>1.10e-1</td>
<td>0.82</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0435</td>
<td>19229</td>
<td>4.40e-2</td>
<td>1.00</td>
<td>2.05e-2</td>
<td>1.00</td>
<td>1.32e-2</td>
<td>1.03</td>
<td>6.62e-2</td>
<td>0.78</td>
</tr>
<tr>
<td>0.0303</td>
<td>39469</td>
<td>3.06e-2</td>
<td>1.00</td>
<td>1.43e-2</td>
<td>1.00</td>
<td>9.19e-3</td>
<td>1.01</td>
<td>5.04e-2</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>0.0217</td>
<td>76545</td>
<td>2.20e-2</td>
<td>1.00</td>
<td>1.02e-2</td>
<td>1.00</td>
<td>6.57e-3</td>
<td>1.01</td>
<td>3.94e-2</td>
<td>0.74</td>
<td></td>
</tr>
<tr>
<td>0.1667</td>
<td>5281</td>
<td>2.86e-3</td>
<td>—</td>
<td>2.20e-3</td>
<td>—</td>
<td>1.78e-3</td>
<td>—</td>
<td>4.47e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0833</td>
<td>20929</td>
<td>9.52e-4</td>
<td>1.62</td>
<td>5.49e-4</td>
<td>2.00</td>
<td>5.30e-4</td>
<td>1.62</td>
<td>2.82e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0435</td>
<td>76545</td>
<td>3.21e-4</td>
<td>1.64</td>
<td>1.49e-4</td>
<td>2.00</td>
<td>1.64</td>
<td>1.82e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0303</td>
<td>157345</td>
<td>1.77e-4</td>
<td>1.65</td>
<td>7.26e-5</td>
<td>2.00</td>
<td>1.10e-4</td>
<td>1.65</td>
<td>1.43e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0217</td>
<td>305441</td>
<td>1.02e-4</td>
<td>1.65</td>
<td>3.74e-5</td>
<td>2.00</td>
<td>6.37e-5</td>
<td>1.65</td>
<td>1.15e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.1667</td>
<td>10945</td>
<td>4.95e-4</td>
<td>1.66</td>
<td>1.52e-4</td>
<td>—</td>
<td>3.30e-4</td>
<td>—</td>
<td>2.57e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0833</td>
<td>43489</td>
<td>1.56e-4</td>
<td>1.67</td>
<td>2.40e-4</td>
<td>2.67</td>
<td>1.04e-4</td>
<td>1.67</td>
<td>1.62e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0435</td>
<td>159253</td>
<td>5.28e-5</td>
<td>1.67</td>
<td>4.23e-7</td>
<td>2.67</td>
<td>3.51e-5</td>
<td>1.67</td>
<td>1.05e-2</td>
<td>0.67</td>
</tr>
<tr>
<td>0.0303</td>
<td>327493</td>
<td>2.89e-5</td>
<td>1.67</td>
<td>1.49e-4</td>
<td>2.00</td>
<td>2.00e-4</td>
<td>1.67</td>
<td>8.24e-3</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0217</td>
<td>635905</td>
<td>1.02e-5</td>
<td>1.67</td>
<td>1.62e-5</td>
<td>2.00</td>
<td>1.10e-4</td>
<td>1.67</td>
<td>6.61e-3</td>
<td>0.67</td>
<td></td>
</tr>
</tbody>
</table>

Table 8. Example 2, refinement with quadrilaterals and using the $L^2$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$\gamma(\sigma)$</th>
<th>$e(u)$</th>
<th>$\gamma(u)$</th>
<th>$e(p)$</th>
<th>$\gamma(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$\gamma(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1667</td>
<td>1825</td>
<td>2.14e-1</td>
<td>—</td>
<td>7.92e-2</td>
<td>—</td>
<td>1.05e-1</td>
<td>—</td>
<td>2.49e-1</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0927</td>
<td>5985</td>
<td>8.75e-2</td>
<td>1.52</td>
<td>4.03e-2</td>
<td>1.15</td>
<td>3.59e-2</td>
<td>1.83</td>
<td>1.23e-1</td>
<td>1.20</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0478</td>
<td>22533</td>
<td>3.83e-2</td>
<td>1.25</td>
<td>1.98e-2</td>
<td>1.08</td>
<td>1.24e-2</td>
<td>1.61</td>
<td>6.91e-2</td>
<td>0.87</td>
</tr>
<tr>
<td>0.0321</td>
<td>49665</td>
<td>2.47e-2</td>
<td>1.10</td>
<td>1.31e-2</td>
<td>1.04</td>
<td>7.26e-3</td>
<td>1.34</td>
<td>5.13e-2</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>0.0239</td>
<td>89441</td>
<td>1.81e-2</td>
<td>1.06</td>
<td>9.67e-3</td>
<td>1.03</td>
<td>5.07e-3</td>
<td>1.22</td>
<td>4.16e-2</td>
<td>0.72</td>
<td></td>
</tr>
<tr>
<td>0.1667</td>
<td>6241</td>
<td>3.92e-3</td>
<td>—</td>
<td>1.81e-3</td>
<td>—</td>
<td>2.68e-3</td>
<td>—</td>
<td>7.13e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0927</td>
<td>20681</td>
<td>1.20e-3</td>
<td>1.81</td>
<td>1.81e-3</td>
<td>—</td>
<td>6.73e-3</td>
<td>—</td>
<td>7.13e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0478</td>
<td>78347</td>
<td>3.66e-4</td>
<td>1.79</td>
<td>1.07e-4</td>
<td>2.17</td>
<td>2.54e-4</td>
<td>1.79</td>
<td>2.73e-2</td>
<td>0.72</td>
</tr>
<tr>
<td>0.0321</td>
<td>173057</td>
<td>1.86e-4</td>
<td>1.71</td>
<td>4.70e-5</td>
<td>2.09</td>
<td>1.29e-4</td>
<td>1.70</td>
<td>2.08e-2</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>0.0239</td>
<td>321009</td>
<td>1.13e-4</td>
<td>1.69</td>
<td>2.56e-5</td>
<td>2.06</td>
<td>7.88e-5</td>
<td>1.69</td>
<td>1.71e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.1667</td>
<td>12385</td>
<td>9.85e-4</td>
<td>—</td>
<td>1.04e-4</td>
<td>—</td>
<td>6.75e-4</td>
<td>—</td>
<td>4.90e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0927</td>
<td>41185</td>
<td>3.13e-4</td>
<td>1.96</td>
<td>2.03e-6</td>
<td>2.78</td>
<td>2.14e-4</td>
<td>1.96</td>
<td>3.02e-2</td>
<td>0.83</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0478</td>
<td>156349</td>
<td>9.68e-5</td>
<td>1.77</td>
<td>3.87e-5</td>
<td>2.50</td>
<td>6.61e-5</td>
<td>1.77</td>
<td>1.87e-2</td>
<td>0.72</td>
</tr>
<tr>
<td>0.0321</td>
<td>345601</td>
<td>4.93e-5</td>
<td>1.70</td>
<td>1.41e-5</td>
<td>2.55</td>
<td>3.37e-5</td>
<td>1.70</td>
<td>1.43e-2</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>0.0239</td>
<td>623329</td>
<td>3.00e-5</td>
<td>1.69</td>
<td>6.56e-5</td>
<td>2.59</td>
<td>2.05e-5</td>
<td>1.69</td>
<td>1.17e-2</td>
<td>0.67</td>
<td></td>
</tr>
</tbody>
</table>

$\sigma$, and $\text{div}(\sigma)$. In turn, because of these rates of convergence, we realize that our approach should certainly be strengthened with the further incorporation of an adaptive strategy based on a proper a posteriori error estimator. This issue will be addressed in a separate work. Finally, some components of the approximate solutions are displayed in Figs. 3 and 4. Once again, we use $k = 2$, the second mesh of each decomposition and the $L^2$-projection.

We end this paper by remarking that the numerical examples presented in this section confirm the suitability of our mixed virtual element scheme, based on either the $L^2$ or the CG projection, to solve the Brinkman problem (2.1). Now, regarding
the choice of one or the other, we first remark that both projectors yield the same approximation property. Secondly, on one hand the $L^2$-projector is certainly more general, whereas on the other hand the $CG$ one is more ad hoc to the Brinkman problem since it maps $H(div; K)$ into a polynomial space $\hat{H}^1_K$ that resembles the main identity satisfied by the continuous solution, namely $\sigma d = \mu \nabla u \in L^2(\Omega)$ with $\text{div}(u) = 0$. Consequently, as already observed in Sec. 4.4, and on the contrary to the $L^2$ one, the evaluation of the $CG$-projector in the elements of the local virtual spaces does not make use of the degrees of freedom determined by the basis $G_{\perp}^k(K)$ (cf. (3.6)), which, yielding a simpler implementation, is particularly important when obtaining the computable approximations of the continuous solution (cf. Sec. 5.2).

Table 9. Example 2, refinement with hexagons and using the $L^2$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$r(\sigma)$</th>
<th>$e(u)$</th>
<th>$r(u)$</th>
<th>$e(p)$</th>
<th>$r(p)$</th>
<th>$e(\sigma^\star)$</th>
<th>$r(\sigma^\star)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0672</td>
<td>6933</td>
<td>7.86e-2</td>
<td>—</td>
<td>4.09e-2</td>
<td>—</td>
<td>2.98e-2</td>
<td>—</td>
<td>1.17e-1</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0385</td>
<td>19113</td>
<td>4.65e-2</td>
<td>0.94</td>
<td>2.46e-2</td>
<td>0.91</td>
<td>1.27e-2</td>
<td>1.06</td>
<td>8.04e-2</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>0.0170</td>
<td>97173</td>
<td>2.04e-2</td>
<td>1.00</td>
<td>1.09e-2</td>
<td>1.00</td>
<td>5.33e-2</td>
<td>1.03</td>
<td>4.51e-2</td>
<td>0.70</td>
<td></td>
</tr>
<tr>
<td>0.0672</td>
<td>20795</td>
<td>1.15e-3</td>
<td>—</td>
<td>5.24e-4</td>
<td>—</td>
<td>7.91e-4</td>
<td>—</td>
<td>4.65e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0385</td>
<td>57335</td>
<td>4.92e-4</td>
<td>1.52</td>
<td>1.89e-4</td>
<td>1.83</td>
<td>3.40e-4</td>
<td>1.52</td>
<td>3.29e-2</td>
<td>0.62</td>
<td></td>
</tr>
<tr>
<td>0.0275</td>
<td>112019</td>
<td>2.83e-4</td>
<td>1.64</td>
<td>9.67e-5</td>
<td>1.99</td>
<td>1.96e-4</td>
<td>1.64</td>
<td>2.23e-2</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>0.0214</td>
<td>184847</td>
<td>1.86e-4</td>
<td>1.67</td>
<td>5.86e-5</td>
<td>1.99</td>
<td>1.29e-4</td>
<td>1.67</td>
<td>2.23e-2</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>0.0170</td>
<td>291515</td>
<td>1.28e-4</td>
<td>1.64</td>
<td>3.70e-5</td>
<td>2.00</td>
<td>8.81e-5</td>
<td>1.64</td>
<td>1.91e-2</td>
<td>0.67</td>
<td></td>
</tr>
</tbody>
</table>

Table 10. Example 2, refinement with triangles and using the $CG$-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$r(\sigma)$</th>
<th>$e(u)$</th>
<th>$r(u)$</th>
<th>$e(p)$</th>
<th>$r(p)$</th>
<th>$e(\sigma^\star)$</th>
<th>$r(\sigma^\star)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1667</td>
<td>1345</td>
<td>1.70e-1</td>
<td>—</td>
<td>7.89e-2</td>
<td>—</td>
<td>5.47e-2</td>
<td>—</td>
<td>1.95e-1</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0833</td>
<td>5281</td>
<td>8.45e-2</td>
<td>1.01</td>
<td>3.93e-2</td>
<td>1.01</td>
<td>2.59e-2</td>
<td>1.08</td>
<td>1.10e-1</td>
<td>0.82</td>
<td></td>
</tr>
<tr>
<td>0.0435</td>
<td>19229</td>
<td>4.00e-2</td>
<td>1.00</td>
<td>2.05e-2</td>
<td>1.00</td>
<td>6.62e-2</td>
<td>1.03</td>
<td>1.94e-2</td>
<td>0.82</td>
<td></td>
</tr>
<tr>
<td>0.0303</td>
<td>39469</td>
<td>3.06e-2</td>
<td>1.00</td>
<td>1.43e-2</td>
<td>1.00</td>
<td>9.19e-3</td>
<td>1.01</td>
<td>5.04e-2</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>0.0217</td>
<td>76545</td>
<td>2.20e-2</td>
<td>1.00</td>
<td>1.02e-2</td>
<td>1.00</td>
<td>6.57e-3</td>
<td>1.01</td>
<td>3.94e-2</td>
<td>0.74</td>
<td></td>
</tr>
<tr>
<td>0.1667</td>
<td>5281</td>
<td>2.86e-3</td>
<td>—</td>
<td>2.20e-3</td>
<td>—</td>
<td>1.79e-3</td>
<td>—</td>
<td>4.47e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0833</td>
<td>157345</td>
<td>1.62</td>
<td>5.49e-4</td>
<td>2.00</td>
<td>5.81e-4</td>
<td>1.62</td>
<td>2.82e-2</td>
<td>0.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0435</td>
<td>76545</td>
<td>3.20e-4</td>
<td>1.64</td>
<td>1.49e-4</td>
<td>2.00</td>
<td>2.00e-4</td>
<td>1.64</td>
<td>1.82e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0303</td>
<td>157345</td>
<td>1.62</td>
<td>5.49e-4</td>
<td>2.00</td>
<td>5.81e-4</td>
<td>1.62</td>
<td>2.82e-2</td>
<td>0.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0217</td>
<td>305441</td>
<td>1.02e-4</td>
<td>1.65</td>
<td>3.74e-4</td>
<td>2.00</td>
<td>6.37e-5</td>
<td>1.65</td>
<td>1.15e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.1667</td>
<td>10945</td>
<td>6.34e-4</td>
<td>—</td>
<td>2.65e-4</td>
<td>—</td>
<td>4.33e-4</td>
<td>—</td>
<td>2.57e-2</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0.0833</td>
<td>343489</td>
<td>2.00e-4</td>
<td>1.67</td>
<td>4.18e-6</td>
<td>2.67</td>
<td>1.36e-4</td>
<td>1.67</td>
<td>1.62e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0435</td>
<td>159253</td>
<td>6.75e-5</td>
<td>1.67</td>
<td>7.38e-7</td>
<td>2.67</td>
<td>4.61e-5</td>
<td>1.67</td>
<td>1.05e-2</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0303</td>
<td>327493</td>
<td>3.79e-5</td>
<td>1.67</td>
<td>2.82e-7</td>
<td>2.67</td>
<td>2.53e-5</td>
<td>1.67</td>
<td>8.24e-3</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>0.0217</td>
<td>635905</td>
<td>2.13e-5</td>
<td>1.67</td>
<td>1.16e-7</td>
<td>2.67</td>
<td>4.15e-5</td>
<td>1.67</td>
<td>6.61e-3</td>
<td>0.67</td>
<td></td>
</tr>
</tbody>
</table>
Table 11. Example 2, refinement with quadrilaterals and using the CG-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$r(\sigma)$</th>
<th>$e(u)$</th>
<th>$r(u)$</th>
<th>$e(p)$</th>
<th>$r(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$r(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1667</td>
<td>1825</td>
<td>2.14e-1</td>
<td>—</td>
<td>7.92e-2</td>
<td>—</td>
<td>1.05e-1</td>
<td>—</td>
<td>2.49e-1</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.0927</td>
<td>5985</td>
<td>8.75e-2</td>
<td>1.52</td>
<td>4.03e-2</td>
<td>1.15</td>
<td>3.59e-2</td>
<td>1.83</td>
<td>1.23e-1</td>
<td>1.20</td>
<td>1.20</td>
</tr>
<tr>
<td>0.0478</td>
<td>22533</td>
<td>3.83e-2</td>
<td>1.25</td>
<td>1.98e-2</td>
<td>1.08</td>
<td>1.24e-2</td>
<td>1.61</td>
<td>0.91e-2</td>
<td>0.87</td>
<td>0.87</td>
</tr>
<tr>
<td>0.0321</td>
<td>49665</td>
<td>2.47e-2</td>
<td>1.10</td>
<td>1.31e-2</td>
<td>1.04</td>
<td>7.26e-3</td>
<td>1.34</td>
<td>5.13e-2</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>0.0239</td>
<td>89441</td>
<td>1.81e-2</td>
<td>1.06</td>
<td>9.67e-3</td>
<td>1.03</td>
<td>5.07e-3</td>
<td>1.22</td>
<td>4.16e-2</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>0.1667</td>
<td>6241</td>
<td>3.92e-3</td>
<td>—</td>
<td>1.81e-3</td>
<td>—</td>
<td>2.72e-3</td>
<td>—</td>
<td>7.13e-2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.0927</td>
<td>20681</td>
<td>1.21e-3</td>
<td>2.01</td>
<td>4.54e-4</td>
<td>2.36</td>
<td>8.43e-4</td>
<td>2.00</td>
<td>4.39e-2</td>
<td>0.83</td>
<td>0.83</td>
</tr>
<tr>
<td>1</td>
<td>0.0478</td>
<td>78347</td>
<td>3.67e-4</td>
<td>1.80</td>
<td>1.07e-4</td>
<td>2.17</td>
<td>2.57e-4</td>
<td>1.79</td>
<td>2.73e-2</td>
<td>0.72</td>
</tr>
<tr>
<td>0.0321</td>
<td>173057</td>
<td>1.86e-4</td>
<td>1.71</td>
<td>4.70e-5</td>
<td>2.09</td>
<td>1.30e-4</td>
<td>1.71</td>
<td>2.08e-2</td>
<td>0.68</td>
<td>0.68</td>
</tr>
<tr>
<td>0.0239</td>
<td>312009</td>
<td>1.13e-4</td>
<td>1.69</td>
<td>2.56e-5</td>
<td>2.06</td>
<td>7.93e-5</td>
<td>1.69</td>
<td>1.71e-2</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>0.1667</td>
<td>12385</td>
<td>9.83e-2</td>
<td>—</td>
<td>7.12e-6</td>
<td>—</td>
<td>6.82e-4</td>
<td>—</td>
<td>4.90e-2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.0927</td>
<td>41185</td>
<td>3.18e-4</td>
<td>1.92</td>
<td>1.45e-6</td>
<td>2.72</td>
<td>2.21e-4</td>
<td>1.92</td>
<td>3.02e-2</td>
<td>0.83</td>
<td>0.83</td>
</tr>
<tr>
<td>2</td>
<td>0.0478</td>
<td>156349</td>
<td>1.01e-4</td>
<td>1.73</td>
<td>2.56e-7</td>
<td>2.61</td>
<td>7.03e-5</td>
<td>1.73</td>
<td>1.87e-2</td>
<td>0.72</td>
</tr>
<tr>
<td>0.0321</td>
<td>345601</td>
<td>5.21e-5</td>
<td>1.67</td>
<td>9.08e-8</td>
<td>2.61</td>
<td>3.62e-5</td>
<td>1.68</td>
<td>1.43e-2</td>
<td>0.68</td>
<td>0.68</td>
</tr>
<tr>
<td>0.0239</td>
<td>623329</td>
<td>3.19e-5</td>
<td>1.67</td>
<td>4.19e-8</td>
<td>2.63</td>
<td>2.21e-5</td>
<td>1.67</td>
<td>1.17e-2</td>
<td>0.67</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Table 12. Example 2, refinement with hexagons and using the CG-projection.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$N$</th>
<th>$e(\sigma)$</th>
<th>$r(\sigma)$</th>
<th>$e(u)$</th>
<th>$r(u)$</th>
<th>$e(p)$</th>
<th>$r(p)$</th>
<th>$e(\sigma^*)$</th>
<th>$r(\sigma^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0672</td>
<td>6933</td>
<td>7.86e-2</td>
<td>—</td>
<td>4.09e-2</td>
<td>—</td>
<td>2.29e-2</td>
<td>—</td>
<td>1.17e-1</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.0385</td>
<td>19113</td>
<td>4.65e-2</td>
<td>0.94</td>
<td>2.46e-2</td>
<td>0.91</td>
<td>1.27e-2</td>
<td>1.06</td>
<td>8.04e-2</td>
<td>0.68</td>
<td>0.68</td>
</tr>
<tr>
<td>0.0275</td>
<td>37341</td>
<td>3.31e-2</td>
<td>1.01</td>
<td>1.76e-2</td>
<td>0.99</td>
<td>8.79e-3</td>
<td>1.08</td>
<td>6.34e-2</td>
<td>0.71</td>
<td>0.71</td>
</tr>
<tr>
<td>0.0214</td>
<td>61617</td>
<td>2.57e-2</td>
<td>1.01</td>
<td>1.37e-2</td>
<td>0.99</td>
<td>6.76e-3</td>
<td>1.05</td>
<td>5.30e-2</td>
<td>0.71</td>
<td>0.71</td>
</tr>
<tr>
<td>0.0170</td>
<td>97173</td>
<td>2.04e-2</td>
<td>1.00</td>
<td>1.09e-2</td>
<td>1.00</td>
<td>5.33e-3</td>
<td>1.03</td>
<td>4.51e-2</td>
<td>0.70</td>
<td>0.70</td>
</tr>
<tr>
<td>0.0672</td>
<td>20795</td>
<td>1.15e-3</td>
<td>—</td>
<td>5.24e-4</td>
<td>—</td>
<td>7.96e-4</td>
<td>—</td>
<td>4.65e-2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.0385</td>
<td>57335</td>
<td>4.95e-4</td>
<td>1.52</td>
<td>1.89e-4</td>
<td>1.83</td>
<td>3.42e-4</td>
<td>1.52</td>
<td>3.29e-2</td>
<td>0.62</td>
<td>0.62</td>
</tr>
<tr>
<td>1</td>
<td>0.0275</td>
<td>112019</td>
<td>2.85e-4</td>
<td>1.64</td>
<td>9.67e-5</td>
<td>1.99</td>
<td>1.57e-4</td>
<td>1.64</td>
<td>2.64e-2</td>
<td>0.65</td>
</tr>
<tr>
<td>0.0214</td>
<td>184847</td>
<td>1.87e-4</td>
<td>1.67</td>
<td>5.86e-5</td>
<td>1.99</td>
<td>1.29e-4</td>
<td>1.67</td>
<td>2.23e-2</td>
<td>0.68</td>
<td>0.68</td>
</tr>
<tr>
<td>0.0170</td>
<td>291515</td>
<td>1.28e-4</td>
<td>1.64</td>
<td>3.70e-5</td>
<td>2.00</td>
<td>8.87e-5</td>
<td>1.64</td>
<td>1.91e-2</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>0.0672</td>
<td>39277</td>
<td>3.75e-4</td>
<td>—</td>
<td>1.21e-6</td>
<td>—</td>
<td>2.62e-4</td>
<td>—</td>
<td>3.23e-2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.0385</td>
<td>108297</td>
<td>1.59e-4</td>
<td>1.54</td>
<td>3.17e-7</td>
<td>2.44</td>
<td>1.11e-4</td>
<td>1.54</td>
<td>2.28e-2</td>
<td>0.62</td>
<td>0.62</td>
</tr>
<tr>
<td>2</td>
<td>0.0275</td>
<td>211589</td>
<td>9.13e-5</td>
<td>1.65</td>
<td>1.30e-7</td>
<td>2.65</td>
<td>6.38e-5</td>
<td>1.65</td>
<td>1.84e-2</td>
<td>0.65</td>
</tr>
<tr>
<td>0.0214</td>
<td>349153</td>
<td>5.98e-5</td>
<td>1.69</td>
<td>6.84e-8</td>
<td>2.55</td>
<td>4.17e-5</td>
<td>1.69</td>
<td>1.56e-2</td>
<td>0.68</td>
<td>0.68</td>
</tr>
<tr>
<td>0.0170</td>
<td>550637</td>
<td>4.09e-5</td>
<td>1.65</td>
<td>3.65e-8</td>
<td>2.73</td>
<td>2.85e-5</td>
<td>1.65</td>
<td>1.33e-2</td>
<td>0.67</td>
<td>0.67</td>
</tr>
</tbody>
</table>

In turn, as mentioned in Sec. 6 of Ref. 10, and differently from the CG one, there is no contribution from the $L^2$-projector to the internal degrees of freedom in $S^K$ (cf. (4.7)), and hence the evaluation of this stabilizing bilinear form becomes simpler as well. According to the above discussion, both projectors, having advantages and disadvantages, are indeed comparable. Nevertheless, if we were forced to choose one of them for the present formulation of the Brinkman problem, we would most likely prefer the CG-projector mainly because the dimension of the polynomial space where it projects is strictly less than the dimension of the space where the $L^2$ one projects (cf. (4.13) and (4.15) in Sec. 4.4).
Fig. 3. Example 2, $\sigma_{h,12}$ (top), $\sigma_{h,22}$ (center) and $u_{h,2}$ (bottom), using $k = 2$ and the second mesh of each kind (columns).
Fig. 4. Example 2, $\sigma_{12}^\epsilon$ (top), $\sigma_{22}^\epsilon$ (center) and $p_h$ (bottom), using $k = 2$ and the second mesh of each kind (columns).
Acknowledgments

This work was partially supported by CONICYT-Chile through BASAL project CMM, Universidad de Chile, project Anillo ACT1118 (ANANUM), and the Becas-CONICYT Programme for foreign students; by Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción; and by Universidad Nacional (Costa Rica), through the project 0106-16.

References


17. E. Cáceres, Mixed virtual element methods. Applications in fluid mechanics, Thesis leading to the professional title of Mathematical Civil Engineer, Universidad de Concepción, Chile (2015).